.

Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations in Multi-dimensions

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Dedicated to Ryan Sayers (1982-2003)

Abstract

A direct method for the computation of polynomial conservation laws of polynomial systems of nonlinear partial differential equations (PDEs) in multi-dimensions is presented. The method avoids advanced differential-geometric tools. Instead, it is solely based on calculus, variational calculus, and linear algebra.

Densities are constructed as linear combinations of scaling homogeneous terms with undetermined coefficients. The variational derivative (Euler operator) is used to compute the undetermined coefficients. The homotopy operator is used to compute the fluxes.

The method is illustrated with nonlinear PDEs describing wave phenomena in fluid dynamics, plasma physics, and quantum physics. For PDEs with parameters, the method determines the conditions on the parameters so that a sequence of conserved densities might exist. The existence of a large number of conservation laws is a predictor for complete integrability. The method is algorithmic, applicable to a variety of PDEs, and can be implemented in computer algebra systems such as *Mathematica*, *Maple*, and *REDUCE*.

1 Introduction

Nonlinear partial differential equations (PDEs) that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydro-dynamics, nonlinear optics, and the bio-sciences. Conservation laws are fundamental laws of physics. They maintain that a certain quantity, e.g. momentum, mass (matter), electric charge, or energy, will not change with time during physical processes. Often the PDE itself is a conservation law, e.g. the continuity equation relating charge to current.

As shown in [8] and the articles in this issue, computer algebra systems (CAS) like *Mathematica*, *Maple*, and *REDUCE*, are useful to tackle computational problems in chemistry. Finding closed-form conservation laws of nonlinear PDEs is a nice example. Using CAS interactively, we could make a judicious guess (ansatz) and find a few simple densities and fluxes. Yet, that approach is fruitless for complicated systems with nontrivial conservation laws. Furthermore, completely integrable PDEs [1, 2] admit infinitely many independent conservation laws. Computing them is a challenging task. It involves tedious computations which are prone to error if done with pen and paper. The most famous example is the Korteweg-de Vries (KdV) equation from soliton theory [1, 2] which describes water waves in shallow water, ion-acoustic waves in plasmas, etc. Our earlier work [3, 16] dealt primarily with the symbolic computation of conservation laws of completely integrable PDEs in $(1+1)$ dimensions (with independent variables x and t). In this paper we present a symbolic method that covers PDEs in multi-dimensions (e.g. x, y, z, and t), irrespective of their complete integrability. As before, our approach relies on the concept of dilation (scaling) invariance which limits it to polynomial conserved densities of polynomial PDEs in evolution form.

There are many reasons to compute conserved densities and fluxes of PDEs explicitly. Invariants often lead to new discoveries as was the case in soliton theory. We may want to

verify if conserved quantities of physical importance (e.g. momentum, energy, Hamiltonians, entropy, density, charge) are intact after constitutive relations have been added to close a system. For PDEs with arbitrary parameters we may wish to compute conditions on the parameters so that the model admits conserved quantities. Conserved densities also facilitate the study of qualitative properties of PDEs, such as bi- or tri-Hamiltonian structures. They often guide the choice of solution methods or reveal the nature of special solutions. For example, an infinite sequence of conserved densities assures complete integrability [1] of the PDE, i.e. solvability by the Inverse Scattering Transform [1] and the existence of solitons [2].

Conserved densities aid in the design of numerical solvers for PDEs [27]. Indeed, semi-discretizations that conserve discrete conserved quantities lead to stable numerical schemes (i.e. free of nonlinear instabilities and blowup). While solving differential-difference equations (DDEs), which arise in nonlinear lattices and as semi-discretizations of PDEs, one should check that their conserved quantities indeed remain unchanged. Capitalizing on the analogy between PDEs and DDEs, the techniques presented in this paper have been adapted to DDEs [12, 17, 19, 20, 21] and fully discretized lattices [14, 15].

There are various methods (see [19]) to compute conservation laws of nonlinear PDEs. A common approach relies on the link between conservation laws and symmetries as stated in Noether's theorem [6, 22, 26]. However, the computation of generalized (variational) symmetries, though algorithmic, is as daunting a task as the direct computation of conservation laws. Nonetheless, we draw the reader's attention to DE_APPLS, a package for constructing conservation laws from symmetries available within *Vessiot* [7], a general purpose suite of *Maple* packages for computations on jet spaces. Other methods circumvent the existence of a variational principle [4, 5, 29, 30] but still rely on the symbolic solution of a determining system of PDEs. Despite their power, only a few of the above methods have been fully implemented in CAS (see [16, 19, 29]).

We purposely avoid Noether's theorem, pre-knowledge of symmetries, and a Lagrangian formulation. Neither do we use differential forms or advanced differential-geometric tools. Instead, we present and implement our tools in the language of calculus, linear algebra, and variational calculus. Our down-to-earth calculus formulas are transparent, easy to use by scientists and engineers, and are readily adaptable to nonlinear DDEs (not covered in *Vessiot*).

To design a reliable algorithm, we must compute both densities and fluxes. For the latter, we need to invert the divergence operator which requires the integration (by parts) of an expression involving arbitrary functions. That is where the *homotopy operator* comes into play. Indeed, the homotopy operator reduces that problem to a standard one-dimensional integration with respect to a single auxiliary parameter. One of the first[1] uses of the homotopy operator in the context of conservation laws can be found in [5]. The homotopy operator is a universal, yet little known, tool that can be applied to many problems in which integration by parts in multi-variables is needed. A literature search revealed that homotopy operators are used in integrability testing and inversion problems involving PDEs, DDEs, lattices, and beyond [19]. Assuming the reader is unfamiliar with homotopy operators, we

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[1]We refer the reader to [26, p. 374] for a brief history of the homotopy operator.

"demystify" the homotopy formulas and make them ready for use in nonlinear sciences.

The particular application in this paper is computation of conservation laws for which our algorithm proceeds as follows: build a candidate density as a linear combination (with undetermined coefficients) of "building blocks" that are homogeneous under the scaling symmetry of the PDE. If no such symmetry exists, construct one by introducing parameters with scaling. Subsequently, use the conservation equation and the variational derivative to derive a linear algebraic system for the undetermined coefficients. After the system is solved, use the homotopy operator to compute the flux. Implementations for $(1+1)$-dimensional PDEs [16, 17] in *Mathematica* and *Maple* can be downloaded from [10, 18]. *Mathematica* code that automates the computations for PDEs in multi-dimensions is being designed.

This paper is organized as follows. We establish some connections with vector calculus in Section 2. In Section 3 we list the nonlinear PDEs that will be used throughout the paper: the Korteweg-de Vries and Boussinesq equations from soliton theory [1, 2], the Landau-Lifshitz equation for Heisenberg's ferromagnet [13] and a system of shallow water wave equations for ocean waves [11]. Sections 4 and 5 cover the dilation invariance and conservation laws of those four examples. In Section 6, we introduce and apply the tools from the calculus of variations. Formulas for the variational derivative in multi-dimensions are given in Section 6.1, where we apply the Euler operator for testing the exactness of various expressions. Removal of divergence-equivalent terms with the Euler operator is discussed in Section 6.2. The higher Euler operators and homotopy operators in 1D and 2D are in Sections 6.3 and 6.4. In the latter section, we apply the homotopy operator to integrate by parts and to invert of the divergence operator. In Section 7 we present our three-step algorithm to compute conservation laws and apply it to the KdV equation (Section 7.1), the Boussinesq equation (Section 7.2), the Landau-Lifshitz equation (Section 7.3), and the shallow water wave equations (Section 7.4). Conclusions are drawn in Section 8.

2 Connections with Vector Calculus

We address a few issues in multivariate calculus which complement the material in [9]: (i) To determine whether or not a vector field $\mathbf{F}$ is *conservative*, i.e. $\mathbf{F} = \boldsymbol{\nabla} f$ for some scalar field f, one must verify that $\mathbf{F}$ is *irrotational* or *curl free*, that is $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. The cross ($\times$) denotes the Euclidean cross product. The field f can be computed via standard integrations [24, p. 518, 522].

(ii) To test if $\mathbf{F}$ is the curl of some vector field $\mathbf{G}$, one must check that $\mathbf{F}$ is *incompressible* or *divergence free*, i.e. $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. The dot ($\cdot$) denotes the Euclidean inner product. The components of $\mathbf{G}$ result from solving a coupled system of first-order PDEs [24, p. 526].

(iii) To verify whether or not a scalar field f is the divergence of some vector function $\mathbf{F}$, no theorem from vector calculus comes to the rescue. Furthermore, the computation of $\mathbf{F}$ such that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$ is a nontrivial matter which requires the use of the homotopy operator or Hodge decomposition [9]. In single variable calculus, it amounts to computing the primitive $F = \int f \, dx$.

In multivariate calculus, all scalar fields f, including the components F_i of vector fields $\mathbf{F} = (F_1, F_2, F_3)$, are functions of the independent variables (x, y, z). In differential geometry the above issues are addressed in much greater generality. The functions f and F_i can now depend on arbitrary functions $u(x, y, z), v(x, y, z)$, etc. and their mixed derivatives (up to a fixed order) with respect to (x, y, z). Such functions are called *differential functions* [26]. As one might expect, carrying out the gradient-, curl-, or divergence-test requires advanced algebraic machinery. For example, to test whether or not $f = \boldsymbol{\nabla} \cdot \mathbf{F}$ requires the use of the variational derivative (Euler operator) in 2D or 3D as we will show in Section 6.1. The actual computation of $\mathbf{F}$ requires integration by parts. That is where the higher Euler and homotopy operators enter the picture (see Sections 6.3 and 6.4).

At the moment, no major CAS have reliable routines for integrating expressions involving arbitrary functions and their derivatives. As far as we know, CAS offer no functions to test if a differential function is a divergence. Routines to symbolically invert the total divergence are certainly lacking. In Section 6 we will present these tools and apply them in Section 7.

3 Examples of Nonlinear PDEs

Definition: We consider a nonlinear system of *evolution* equations in $(3 + 1)$ dimensions,

$$\mathbf{u}_t = \mathbf{G}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z, \mathbf{u}_{2x}, \mathbf{u}_{2y}, \mathbf{u}_{2z}, \mathbf{u}_{xy}, \mathbf{u}_{xz}, \mathbf{u}_{yz}, \ldots), \tag{1}$$

where $\mathbf{x} = (x, y, z)$ and t are space and time variables. The vector $\mathbf{u}(x, y, z, t)$ has N components u_i. In the examples we denote the components of $\mathbf{u}$ by u, v, w, etc.. Throughout the paper we use the subscript notation for partial derivatives,

$$\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{u}_{2x} = \mathbf{u}_{xx} = \frac{\partial^2 \mathbf{u}}{\partial x^2}, \quad \mathbf{u}_{2x3y} = \mathbf{u}_{xxyyy} = \frac{\partial^5 \mathbf{u}}{\partial x^2 \partial y^3}, \quad \mathbf{u}_{xy} = \frac{\partial^2 \mathbf{u}}{\partial x \partial y}, \quad etc.. \tag{2}$$

We assume that $\mathbf{G}$ is smooth [9] and does not explicitly depend on $\mathbf{x}$ and t. There are no restrictions on the number of components, order, and degree of nonlinearity of $\mathbf{G}$.

We will predominantly work with polynomial systems, although systems involving one transcendental nonlinearity can also be handled [3, 19]. If parameters are present in (1), they will be denoted by lower-case Greek letters. Throughout this paper we work with the following four prototypical PDEs:

Example 1: The ubiquitous Korteweg-de Vries (KdV) equation [1, 2]

$$u_t + uu_x + u_{3x} = 0, \tag{3}$$

for $u(x, t)$ describes unidirectional shallow water waves and ion-acoustic waves in plasmas.

Example 2: The wave equation,

$$u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \tag{4}$$

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for $u(x, t)$ with real parameter α, was proposed by Boussinesq to describe surface waves in shallow water [1]. For what follows, we rewrite (4) as a system of evolution equations,

$$u_t + v_x = 0, \quad v_t + u_x - 3uu_x - \alpha u_{3x} = 0, \tag{5}$$

where $v(x, t)$ is an auxiliary dependent variable.

Example 3: The classical Landau-Lifshitz (LL) equation,

$$\mathbf{S}_t = \mathbf{S} \times \Delta \mathbf{S} + \mathbf{S} \times D\mathbf{S}, \tag{6}$$

without external magnetic field, models nonlinear spin waves in a continuous Heisenberg ferromagnet [13, 28]. The spin vector $\mathbf{S}(x, t)$ has real components $(u(x, t), v(x, t), w(x, t))$; $\Delta = \boldsymbol{\nabla}^2$ is the Laplacian, and $D = \mathrm{diag}(\alpha, \beta, \gamma)$ is a diagonal matrix with real coupling constants α, β, and γ along the downward diagonal. Splitting (6) into components we get

$$\begin{aligned}
u_t &= vw_{2x} - wv_{2x} + (\gamma - \beta)vw, \\
v_t &= wu_{2x} - uw_{2x} + (\alpha - \gamma)uw, \\
w_t &= uv_{2x} - vu_{2x} + (\beta - \alpha)uv.
\end{aligned} \tag{7}$$

The second and third equations can be obtained from the first by *cyclic* permutations

$$(u \to v, v \to w, w \to u) \quad \text{and} \quad (\alpha \to \beta, \beta \to \gamma, \gamma \to \alpha). \tag{8}$$

We take advantage of the cyclic nature of (7) in the computations in Section 7.3.

Example 4: The (2+1)-dimensional shallow-water wave (SWW) equations,

$$\begin{aligned}
\mathbf{u}_t + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} &= -\boldsymbol{\nabla}(h\theta) + \tfrac{1}{2}h\boldsymbol{\nabla}\theta, \\
h_t + \boldsymbol{\nabla} \cdot (h\mathbf{u}) = 0, \quad \theta_t + \mathbf{u} \cdot (\boldsymbol{\nabla}\theta) &= 0,
\end{aligned} \tag{9}$$

describe waves in the ocean using layered models [11]. Vectors $\mathbf{u} = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$ and $\boldsymbol{\Omega} = \Omega\mathbf{k}$ are the fluid and angular velocities. $\boldsymbol{\nabla} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$ is the gradient operator and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the x, y, and z-axes. $\theta(x, y, t)$ is the horizontally varying potential temperature field and $h(x, y, t)$ is the layer depth. System (9) can be written as

$$\begin{aligned}
u_t + uu_x + vu_y - 2\Omega v + \tfrac{1}{2}h\theta_x + \theta h_x &= 0, \; v_t + uv_x + vv_y + 2\Omega u + \tfrac{1}{2}h\theta_y + \theta h_y = 0, \\
h_t + hu_x + uh_x + hv_y + vh_y &= 0, \quad \theta_t + u\theta_x + v\theta_y = 0.
\end{aligned} \tag{10}$$

4 Key Concept: Dilation or Scaling Invariance

Definition: System (1) is *dilation invariant* if it does not change under a scaling symmetry.
Example: The KdV equation (3) is dilation invariant under the scaling symmetry

$$(x, t, u) \to (\lambda^{-1}x, \lambda^{-3}t, \lambda^2 u), \tag{11}$$

where λ is an arbitrary parameter. Indeed, apply the chain rule to verify that λ^5 factors out.

Definition: The *weight*[2], W, of a variable is the exponent in λ^p which multiplies the variable.

Example: We will always replace x by $\lambda^{-1}x$. Thus, $W(x) = -1$ or $W(\partial/\partial x) = 1$. In view of (11), $W(\partial/\partial t) = 3$ and $W(u) = 2$ for the KdV equation (3).

Definition: The *rank* of a monomial is defined as the total weight of the monomial. An expression is *uniform in rank* if its monomial terms have equal rank.

Example: All monomials in (3) have rank 5. Thus, (3) is uniform in rank with respect to (11).

Weights of dependent variables and weights of $\partial/\partial x, \partial/\partial y$, etc. are assumed to be non-negative and rational. Ranks must be positive integer or rational numbers. The ranks of the equations in (1) may differ from each other.

Conversely, requiring uniformity in rank for each equation in (1) we can compute the weights, and thus the scaling symmetry, with linear algebra.

Example: Indeed, for the KdV equation (3) we have

$$W(u) + W(\partial/\partial t) = 2W(u) + 1 = W(u) + 3, \tag{12}$$

where we used $W(\partial/\partial x) = 1$. Solving (12) yields

$$W(u) = 2, \quad W(\partial/\partial t) = 3. \tag{13}$$

This means that x scales with λ^{-1}, t with λ^{-3}, and u with λ^2, as claimed in (11).

Dilation symmetries, which are special Lie-point symmetries [26], are common to many nonlinear PDEs. However, non-uniform PDEs can be made uniform by giving appropriate weights to parameters that appear in the PDEs, or, if necessary, by extending the set of dependent variables with *auxiliary* parameters with appropriate weights. Upon completion of the computations we can set the auxiliary parameters equal to 1.

Example: The Boussinesq system (5) is not uniform in rank because the terms u_x and αu_{3x} lead to a contradiction in the weight equations. To circumvent the problem we introduce an auxiliary parameter β with (unknown) weight, and replace (5) by

$$
\begin{aligned}
u_t + v_x &= 0, \\
v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0.
\end{aligned}
\tag{14}
$$

Requiring uniformity in rank, we obtain (after some algebra)

$$W(u) = 2, \quad W(v) = 3, \quad W(\beta) = 2, \quad W(\frac{\partial}{\partial t}) = 2. \tag{15}$$

Therefore, (14) is invariant under the scaling symmetry

$$(x, t, u, v, \beta) \rightarrow (\lambda^{-1}x, \lambda^{-2}t, \lambda^2 u, \lambda^3 v, \lambda^2 \beta). \tag{16}$$

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[2]The weights are the remnants of physical units after non-dimensionalization of the PDE.

Definition: System (1) is called *multi-uniform* in rank if it admits more than one scaling symmetry (none of which result from introducing unnecessary parameters with weights).

Example: The LL system (7) is not uniform in rank unless we allow the parameters α, β, and γ to have weights. Doing so, uniformity in rank requires

$$W(u)+W(\frac{\partial}{\partial t}) = W(v)+W(w)+2 = W(\gamma)+W(v)+W(w) = W(\beta)+W(v)+W(w),$$

$$W(v)+W(\frac{\partial}{\partial t}) = W(u)+W(w)+2 = W(\alpha)+W(u)+W(w) = W(\gamma)+W(u)+W(w), \quad (17)$$

$$W(w)+W(\frac{\partial}{\partial t}) = W(u)+W(v)+2 = W(\beta)+W(u)+W(v) = W(\alpha)+W(u)+W(v).$$

Solving these relations gives $W(u) = W(v) = W(w)$, $W(\alpha) = W(\beta) = W(\gamma) = 2$, $W(\frac{\partial}{\partial t}) = W(u) + 2$, where $W(u)$ is arbitrary. The LL equation is thus multi-uniform and invariant under the following class of scaling symmetries,

$$(x, t, u, v, w, \alpha, \beta, \gamma) \rightarrow (\lambda^{-1}x, \lambda^{-(a+2)}t, \lambda^a u, \lambda^a v, \lambda^a w, \lambda^2\alpha, \lambda^2\beta, \lambda^2\gamma), \qquad (18)$$

where $W(u) = a$ is an arbitrary non-negative integer or rational number. For example, if we take $a = 1$ then $W(u) = W(v) = W(w) = 1$, $W(\alpha) = W(\beta) = W(\gamma) = 2$, $W(\frac{\partial}{\partial t}) = 3$ and (7) is invariant under $(x, t, u, v, w, \alpha, \beta, \gamma) \rightarrow (\lambda^{-1}x, \lambda^{-3}t, \lambda u, \lambda v, \lambda w, \lambda^2\alpha, \lambda^2\beta, \lambda^2\gamma)$. Taking different choices for $a < W(\alpha) = 2$ will prove advantageous for the computations in Section 7.3.

Example: The SWW equations (10) are not uniform in rank unless we give a weight to Ω. Indeed, uniformity in rank for (10) requires, after some algebra, that

$$W(\partial/\partial t) = W(\Omega), \ W(\partial/\partial y) = W(\partial/\partial x) = 1, \ W(u) = W(v) = W(\Omega) - 1,$$
$$W(\theta) = 2W(\Omega) - W(h) - 2, \qquad (19)$$

where $W(h)$ and $W(\Omega)$ are arbitrary. Hence, the SWW system is multi-uniform and invariant under the class of scaling symmetries,

$$(x, y, t, u, v, h, \theta, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-b}t, \lambda^{b-1}u, \lambda^{b-1}v, \lambda^a h, \lambda^{2b-a-2}\theta, \lambda^b\Omega). \qquad (20)$$

where $W(h) = a$ and $W(\Omega) = b$. Various choices for a and b will aid the computations in Section 7.4.

5   Conservation Laws

Definition: A *conservation law* [26] in differential form is a PDE of the form,

$$D_t\, \rho + \text{Div}\, \mathbf{J} = 0, \qquad (21)$$

which is satisfied on solutions of (1). The total time derivative D_t and total divergence Div are defined below. The scalar differential function ρ is called the *conserved density*;

the vector differential function $\mathbf{J}$ is the associated *flux*. In electromagnetism, (21) is the continuity equation relating *charge density* ρ to *current* $\mathbf{J}$.

In general, ρ and $\mathbf{J}$ are functions of $\mathbf{x}, t, \mathbf{u}$, and the derivatives of $\mathbf{u}$ with respect to the components of $\mathbf{x}$. In this paper, we will assume that densities and fluxes (i) are *local*, which means free of integral terms; (ii) *polynomial* in $\mathbf{u}$ and its derivatives; and (iii) do not explicitly depend on $\mathbf{x}$ and t. So, $\rho(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z, \mathbf{u}_{2x}, \ldots)$ and $\mathbf{J}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z, \mathbf{u}_{2x}, \ldots)$.

Definition: There is a close relationship between conservation laws and *constants of motion*. Indeed, integration of (21) over all space yields the integral form of a conservation law,

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho \, dx \, dy \, dz = \text{constant}, \tag{22}$$

provided that $\mathbf{J}$ vanishes at infinity. As in mechanics, the P's are called constants of motion.

The total divergence Div is computed as $\text{Div} \, \mathbf{J} = (\text{D}_x, \text{D}_y, \text{D}_z) \cdot (J_1, J_2, J_3) = \text{D}_x J_1 + \text{D}_y J_2 + \text{D}_z J_3$. In the 1D case, with one spatial variable (x), conservation equation (21) reduces to

$$\text{D}_t \, \rho + \text{D}_x J = 0, \tag{23}$$

where both density ρ and flux J are scalar differential functions.

The conservation laws (21) and (23) involve total derivatives D_t and D_x. In the 1D case,

$$\text{D}_t \rho = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{n} \frac{\partial \rho}{\partial u_{kx}} \text{D}_x^k(u_t). \tag{24}$$

where n is the order of u in ρ. Upon replacement of u_t, u_{tx}, etc. from $u_t = G$, we get

$$\text{D}_t \rho = \frac{\partial \rho}{\partial t} + \rho(u)'[G], \tag{25}$$

where $\rho(u)'[G]$ is the Fréchet derivative of ρ in the direction of G. Similarly,

$$\text{D}_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{m} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x}, \tag{26}$$

where m is the order of u of J.

Example: Assume that the KdV equation (3) has a density of the form

$$\rho = c_1 u^3 + c_2 u_x^2, \tag{27}$$

where c_1 and c_2 are constants. Since ρ does not explicitly depend on time and is of order $n = 1$ in u, application of (24) gives

$$
\begin{aligned}
\text{D}_t \rho &= \frac{\partial \rho}{\partial u} \text{I}(u_t) + \frac{\partial \rho}{\partial u_x} \text{D}_x(u_t) \\
&= 3c_1 u^2 u_t + 2c_2 u_x u_{tx} = -3c_1 u^2 (u u_x + u_{3x}) - 2c_2 u_x (u u_x + u_{3x})_x \\
&= -3c_1 u^2 (u u_x + u_{3x}) - 2c_2 u_x (u_x^2 + u u_{2x} + u_{4x}) \\
&= -(3c_1 u^3 u_x + 3c_1 u^2 u_{3x} + 2c_2 u_x^3 + 2c_2 u u_x u_{2x} + 2c_2 u_x u_{4x}),
\end{aligned}
\tag{28}
$$

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where $D_x^0 = I$ is the identity operator and where we used (3) to replace u_t and u_{tx}. The generalizations of (24) and (26) to multiple dependent variables is straightforward [26].

Example: Taking $\mathbf{u}(x,t) = (u(x,t), v(x,t), w(x,t))$,

$$D_t\rho = \frac{\partial\rho}{\partial t} + \sum_{k=0}^{n_1}\frac{\partial\rho}{\partial u_{kx}}D_x^k(u_t) + \sum_{k=0}^{n_2}\frac{\partial\rho}{\partial v_{kx}}D_x^k(v_t) + \sum_{k=0}^{n_3}\frac{\partial\rho}{\partial w_{kx}}D_x^k(w_t), \qquad (29)$$

$$D_xJ = \frac{\partial J}{\partial x} + \sum_{k=0}^{m_1}\frac{\partial J}{\partial u_{kx}}u_{(k+1)x} + \sum_{k=0}^{m_2}\frac{\partial J}{\partial v_{kx}}v_{(k+1)x} + \sum_{k=0}^{m_3}\frac{\partial J}{\partial w_{kx}}w_{(k+1)x} \qquad (30)$$

where n_1, n_2 and n_3 are the highest orders of u, v and w in ρ, and m_1, m_2 and m_3 are the highest orders of u, v, w in J.

Example: Assume that the Boussinesq system (14) has a density of the form

$$\rho = c_1\beta^2 u + c_2\beta u^2 + c_3 u^3 + c_4 v^2 + c_5 u_x v + c_6 u_x^2. \qquad (31)$$

where c_1 through c_6 are constants. Since $n_1 = 1$ and $n_2 = 0$, application of (29) gives

$$\begin{aligned}
D_t\rho &= \frac{\partial\rho}{\partial u}I(u_t) + \frac{\partial\rho}{\partial u_x}D_x(u_t) + \frac{\partial\rho}{\partial v}I(v_t) \\
&= (c_1\beta^2 + 2c_2\beta u + 3c_3 u^2)u_t + (c_5 v + 2c_6 u_x)u_{tx} + (2c_4 v + c_5 u_x)v_t \\
&= -\Big((c_1\beta^2 + 2c_2\beta u + 3c_3 u^2)v_x + (c_5 v + 2c_6 u_x)v_{2x} + (2c_4 v + c_5 u_x)(\beta u_x - 3uu_x - \alpha u_{3x})\Big),(32)
\end{aligned}$$

where we replaced u_t, u_{tx}, and v_t from (14).

Remark: The flux $\mathbf{J}$ in (21) is not uniquely defined. In 1D we use (23) and the flux is only determined up to an arbitrary constant. In 2D, the flux is only determined up to a divergence-free vector $\mathbf{K} = (K_1, K_2) = (D_y\phi, -D_x\phi)$, where ϕ is an arbitrary scalar differential function. In 3D, the flux can only be determined up to a curl term. Indeed, if $(\rho, \mathbf{J})$ is a valid density-flux pair, so is $(\rho, \mathbf{J} + \boldsymbol{\nabla} \times \mathbf{K})$ for any arbitrary vector differential function $\mathbf{K} = (K_1, K_2, K_3)$. Recall that $\boldsymbol{\nabla} \times \mathbf{K} = (D_y K_3 - D_z K_2, D_z K_1 - D_x K_3, D_x K_2 - D_y K_1)$.

Example: The Korteweg-de Vries equation (3) is known [1, 2] to have infinitely many polynomial conservation laws. The first three density-flux pairs are

$$\begin{aligned}
\rho^{(1)} &= u, & J^{(1)} &= \frac{1}{2}u^2 + u_{2x}, \\
\rho^{(2)} &= \frac{1}{2}u^2, & J^{(2)} &= \frac{1}{3}u^3 - \frac{1}{2}u_x^2 + uu_{2x}, \\
\rho^{(3)} &= \frac{1}{3}u^3 - u_x^2, & J^{(3)} &= \frac{1}{4}u^4 - 2uu_x^2 + u^2 u_{2x} + u_{2x}^2 - 2u_x u_{3x}.
\end{aligned} \qquad (33)$$

The first two express conservation of momentum and energy, respectively. They are easy to compute by hand. The third one, less obvious and requiring more work, corresponds to Boussinesq's moment of instability [25]. Observe that the above densities are uniform of ranks $2, 4$, and 6. The fluxes are also uniform of ranks $4, 6$, and 8.

The computations get quickly out of hand for higher ranks. For example, for rank 12

$$\rho^{(6)} = \frac{1}{6}u^6 - 10u^3u_x^2 - 5u_x^4 + 18u^2u_{2x}{}^2 + \frac{120}{7}u_{2x}{}^3 - \frac{108}{7}uu_{3x}{}^2 + \frac{36}{7}u_{4x}{}^2 \qquad (34)$$

and $J^{(6)}$ is a scaling homogeneous polynomial with 20 terms of rank 14 (not shown).

In general, if in (23) $\mathrm{rank}(\rho) = R$ then $\mathrm{rank}(J) = R + W(\partial/\partial t) - 1$. All the pieces in (21) are also uniform in rank. This comes as no surprise since the conservation law (21) holds on solutions of (1), hence it 'inherits' the dilation symmetry of (1).

Example: The Boussinesq equation (4) also admits infinitely many conservation laws and is completely integrable [1, 2]. The first four density-flux pairs [3] for (14) are

$$
\begin{aligned}
\rho^{(1)} &= u, & J^{(1)} &= v, \\
\rho^{(2)} &= v, & J^{(2)} &= \beta u - \frac{3}{2}u^2 - \alpha u_{2x}, \\
\rho^{(3)} &= uv, & J^{(3)} &= \frac{1}{2}\beta u^2 - u^3 + \frac{1}{2}v^2 + \frac{1}{2}\alpha u_x^2 - \alpha u u_{2x}, \\
\rho^{(4)} &= \beta u^2 - u^3 + v^2 + \alpha u_x^2, & J^{(4)} &= 2\beta uv - 3u^2 v - 2\alpha u_{2x}v + 2\alpha u_x v_x.
\end{aligned}
\qquad (35)
$$

The densities are of ranks $2, 3, 5$ and 6, respectively. The corresponding fluxes are of one rank higher. After setting $\beta = 1$ we obtain the conserved quantities of (5) even though initially this system was not uniform in rank.

Example: We assume that α, β, and γ in (6) are nonzero. Cases with vanishing parameters must be investigated separately. The first six density-flux pairs [13, 28] are

$$
\begin{aligned}
\rho^{(1)} &= u, & J^{(1)} &= v_x w - v w_x, & (\beta = \gamma \neq \alpha), \\
\rho^{(2)} &= v, & J^{(2)} &= uw_x - u_x w, & (\alpha = \gamma \neq \beta), \\
\rho^{(3)} &= w, & J^{(3)} &= u_x v - u v_x, & (\alpha = \beta \neq \gamma), \\
\rho^{(4)} &= u^2 + v^2 + w^2, & J^{(4)} &= 0, & \\
\rho^{(5)} &= (u^2 + v^2 + w^2)^2, & J^{(5)} &= 0, & \\
\rho^{(6)} &= u_x^2 + v_x^2 + w_x^2 + (\gamma - \alpha)u^2 + (\gamma - \beta)v^2, & & & \\
J^{(6)} &= 2\big((vw_x - v_x w)u_{2x} + (u_x w - u w_x)v_{2x} + (uv_x - u_x v)w_{2x} & & & \\
& \quad + (\beta - \gamma)u_x v w + (\gamma - \alpha)u v_x w + (\alpha - \beta)u v w_x\big). & & &
\end{aligned}
\qquad (36)
$$

Note that the second and third density-flux pairs follow from the first pair via the cyclic permutations in (8). If we select $W(u) = W(v) = W(w) = a = \frac{1}{4}$, then the first three densities have rank $\frac{1}{4}$ and $\rho^{(4)}, \rho^{(5)}$, and $\rho^{(6)}$ have ranks $\frac{1}{2}, 1, \frac{5}{2}$, respectively.

The physics of (6) demand that the magnitude $S = ||\mathbf{S}||$ of the spin field is constant in time. Indeed, using (7) we readily verify that $uu_t + vv_t + ww_t = 0$. Hence, any differentiable function of S is also conserved in time. This is apparent in $\rho^{(4)} = ||\mathbf{S}||^2$ and $\rho^{(5)} = ||\mathbf{S}||^4$, which are conserved (in time) even for the fully anisotropic case where $\alpha \neq \beta \neq \gamma \neq \alpha$.

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Obviously, linear combinations of conserved densities are also conserved. Hence, $\rho^{(6)} = ||\mathbf{S}_x||^2 + (\gamma - \alpha)u^2 + (\gamma - \beta)v^2$ can be replaced by

$$\tilde{\rho}^{(6)} = \frac{1}{2}(\rho^{(6)} - \gamma\rho^{(4)}) = \frac{1}{2}(u_x^2 + v_x^2 + w_x^2 - \alpha u^2 - \beta v^2 - \gamma w^2). \tag{37}$$

Now, $u_x^2 = D_x(uu_x) - uu_{2x}$, etc., and densities are divergence-equivalent (see Section 6.2) if they differ by a total derivative with respect to x. So, (37) is equivalent with

$$\tilde{\rho}^{(6)} = -\frac{1}{2}(uu_{2x} + vv_{2x} + ww_{2x} + \alpha u^2 + \beta v^2 + \gamma w^2), \tag{38}$$

which can be compactly written as

$$\tilde{\rho}^{(6)} = -\frac{1}{2}(\mathbf{S} \cdot \Delta\mathbf{S} + \mathbf{S} \cdot D\mathbf{S}), \tag{39}$$

where Δ is the Laplacian and $D = \mathrm{diag}(\alpha, \beta, \gamma)$. So, $\tilde{\rho}^{(6)}$ is the Hamiltonian density [28] in

$$
\begin{aligned}
H &= -\frac{1}{2}\int (\mathbf{S} \cdot \Delta\mathbf{S} + \mathbf{S} \cdot D\mathbf{S}) \ dx \\
&= \frac{1}{2}\int \left(||\mathbf{S}_x||^2 + (\gamma - \alpha)u^2 + (\gamma - \beta)v^2\right) \ dx.
\end{aligned} \tag{40}
$$

The Hamiltonian H expresses that the total energy is constant in time since $dH/dt = 0$.

Example: The SWW equations (10) admit a Hamiltonian formulation [11] and infinitely many conservation laws. Yet, that is still insufficient to guarantee complete integrability of PDEs in $(2+1)$-dimensions. The first few conserved densities and fluxes for (10) are

$$
\begin{aligned}
&\rho^{(1)} = h, \quad \mathbf{J}^{(1)} = \begin{pmatrix} uh \\ vh \end{pmatrix}, \qquad\quad \rho^{(2)} = h\theta, \quad \mathbf{J}^{(2)} = \begin{pmatrix} uh\theta \\ vh\theta \end{pmatrix}, \\
&\rho^{(3)} = h\theta^2, \quad \mathbf{J}^{(3)} = \begin{pmatrix} uh\theta^2 \\ vh\theta^2 \end{pmatrix}, \\
&\rho^{(4)} = (u^2 + v^2)h + h^2\theta, \qquad\qquad \mathbf{J}^{(4)} = \begin{pmatrix} u^3h + uv^2h + 2uh^2\theta \\ v^3h + u^2vh + 2vh^2\theta \end{pmatrix}, \\
&\rho^{(5)} = (2\Omega - u_y + v_x)\theta, \\
&\mathbf{J}^{(5)} = \tfrac{1}{6}\begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}.
\end{aligned} \tag{41}
$$

As shown in [11], system (10) has conserved densities $\rho = hf(\theta)$ and $\rho = (2\Omega - u_y + v_x)g(\theta)$, where $f(\theta)$ and $g(\theta)$ are arbitrary functions. Such densities are the integrands of the Casimirs of the Poisson bracket associated with (10). The algorithm presented in Section 7.4 only computes densities of the form $\rho = h\theta^k$ and $\rho = (2\Omega - u_y + v_x)\theta^l$, where k and l are positive integers.

6   Tools from the Calculus of Variations

In this section we introduce the variational derivative (Euler operator), the higher Euler operators (also called Lie-Euler operators) from the calculus of variations, and the homotopy operator from homological algebra and variational bi-complexes [6]. These tools will be applied in Section 7.

6.1   Variational Derivative (Euler Operator)

Definition: A scalar differential function f is a *divergence* if and only if there exists a vector differential function $\mathbf{F}$ such that $f = \mathrm{Div}\,\mathbf{F}$. In 1D, we say that a differential function f is *exact*[3] if and only if there exists a scalar differential function F such that $f = \mathrm{D}_x F$. Obviously, $F = \mathrm{D}_x^{-1}(f) = \int f\,dx$ is then the primitive (or integral) of f.

Example: Taking $f = -\mathrm{D}_t\rho$ from (28), that is

$$f = 3c_1 u^3 u_x + 2c_2 u_x^3 + 2c_2 u u_x u_{2x} + 3c_1 u^2 u_{3x} + 2c_2 u_x u_{4x}, \tag{42}$$

where[4] $u(x;t)$, we will compute c_1 and c_2 so that f is exact.

Upon repeated integration by parts (by hand), we get

$$F = \int f\,dx = \frac{3}{4}c_1 u^4 + (c_2 - 3c_1)uu_x^2 + 3c_1 u^2 u_{2x} - c_2 u_{2x}^2 + 2c_2 u_x u_{3x} + \int (3c_1 + c_2)u_x^3\,dx. \tag{43}$$

Removing the "obstruction" (the last term) requires $c_2 = -3c_1$. Substituting a solution, $c_1 = \frac{1}{3}, c_2 = -1$, into (27) gives $\rho^{(3)}$ in (33). Substituting that solution into (42) and (43) yields

$$f = u^3 u_x - 2u_x^3 - 2u u_x u_{2x} + u^2 u_{3x} - 2u_x u_{4x} \tag{44}$$

which is exact, together with its integral

$$F = \frac{1}{4}u^4 - 2uu_x^2 + u^2 u_{2x} + u_{2x}^2 - 2u_x u_{3x}, \tag{45}$$

which is $J^{(3)}$ in (33). Currently, CAS like *Mathematica*, *Maple*[5], and *REDUCE* cannot compute (43) as a sum of terms that can be integrated out and the obstruction.

Example: Consider the following example in 2D

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x, \tag{46}$$

where $u(x,y)$ and $v(x,y)$. It is easy to verify by hand (via integration by parts) that f is exact. Indeed, $f = \mathrm{Div}\,\mathbf{F}$ with

$$\mathbf{F} = (uv_y - u_x v_y, -uv_x + u_x v_x). \tag{47}$$

---

[3]We do not use *integrable* to avoid confusion with complete integrability from soliton theory [1, 2].

[4]Variable t is parameter in subsequent computations. For brevity, we write $u(x)$ instead of $u(x;t)$.

[5]Future versions of *Maple* will be able to "partially" integrate such expressions [10, 19].

As far as we know, the leading CAS currently lack tools to compute $\mathbf{F}$. Two questions arise:
(i) How can we tell whether f is the divergence of some differential function $\mathbf{F}$?
(ii) How do we compute $\mathbf{F}$ automatically and without multivariate integration by parts of expressions involving arbitrary functions?

To answer these questions we use the following tools from the calculus of variations: the variational derivative (Euler operator), its generalizations (higher Euler operators or Lie-Euler operators), and the homotopy operator.

Definition: The *variational derivative* (zeroth Euler operator), $\mathcal{L}_{\mathbf{u}(\mathbf{x})}^{(\mathbf{0})}$, is defined [26, p. 246] by

$$\mathcal{L}_{\mathbf{u}(\mathbf{x})}^{(\mathbf{0})} = \sum_{J}(-\mathrm{D})_J \frac{\partial}{\partial \mathbf{u}_J}, \tag{48}$$

where the sum is over all the unordered multi-indices J [26, p. 95]. For example, in the 2D case the multi-indices corresponding to second-order derivatives can be identified with $\{2x, 2y, 2z, xy, xz, yz\}$. Obviously, $(-D)_{2x} = (-\mathrm{D}_x)^2 = \mathrm{D}_x^2$, $(-D)_{xy} = (-\mathrm{D}_x)(-\mathrm{D}_y) = \mathrm{D}_x\mathrm{D}_y$, etc.. For notational details see [26, p. 95, p. 108, p. 246].

With applications in mind, we give calculus-based formulas for the variational derivatives in 1D and 2D. The formula for 3D is analogous and can be found in [19].

Example: In 1D, where $\mathbf{x} = x$, for scalar component $u(x)$ we have[6]

$$\mathcal{L}_{u(x)}^{(0)} = \sum_{k=0}^{\infty}(-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}} = \frac{\partial}{\partial u} - \mathrm{D}_x\frac{\partial}{\partial u_x} + \mathrm{D}_x^2\frac{\partial}{\partial u_{2x}} - \mathrm{D}_x^3\frac{\partial}{\partial u_{3x}} + \cdots, \tag{49}$$

In 2D where $\mathbf{x} = (x, y)$, we have for component $u(x, y)$

$$\mathcal{L}_{u(x,y)}^{(0,0)} = \sum_{k_x=0}^{\infty}\sum_{k_y=0}^{\infty}(-\mathrm{D}_x)^{k_x}(-\mathrm{D}_y)^{k_y}\frac{\partial}{\partial u_{k_x x\, k_y y}} = \frac{\partial}{\partial u} - \mathrm{D}_x\frac{\partial}{\partial u_x} - \mathrm{D}_y\frac{\partial}{\partial u_y}$$
$$+ \mathrm{D}_x^2\frac{\partial}{\partial u_{2x}} + \mathrm{D}_x\mathrm{D}_y\frac{\partial}{\partial u_{xy}} + \mathrm{D}_y^2\frac{\partial}{\partial u_{2y}} - \mathrm{D}_x^3\frac{\partial}{\partial u_{3x}} - \cdots, \tag{50}$$

Note that $u_{k_x x\, k_y y}$ stands for $u_{xx\cdots x\, yy\cdots y}$ where x is repeated k_x times and y is repeated k_y times. Similar formulas hold for components v, w, etc.. The first question is then answered by the following theorem [26, p. 248].

Theorem: A necessary and sufficient condition for a function f to be a divergence, i.e. there exists a differential function $\mathbf{F}$ so that $f = \mathrm{Div}\,\mathbf{F}$, is that $\mathcal{L}_{\mathbf{u}(\mathbf{x})}^{(\mathbf{0})}(f) \equiv 0$. In words: the Euler operator annihilates divergences.

If, for example, $\mathbf{u}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x}))$ then both $\mathcal{L}_{u(\mathbf{x})}^{(\mathbf{0})}(f)$ and $\mathcal{L}_{v(\mathbf{x})}^{(\mathbf{0})}(f)$ must vanish identically. For the 1D case, the theorem says that a differential function f is exact, i.e. there exists a differential function F so that $f = D_xF$, if and only if $\mathcal{L}_{\mathbf{u}(x)}^{(\mathbf{0})}(f) \equiv 0$.

---

[6]Variable t in $\mathbf{u}(\mathbf{x}; t)$ is suppressed because it is a parameter in the Euler operators.

Example: Avoiding integration by parts, we determine c_1 and c_2 so that f in (42) will be exact. Since f is of order 4 in the (one) dependent variable $u(x)$, the zeroth Euler operator (49) terminates at $k = 4$. Using nothing but differentiations, we readily compute

$$
\begin{aligned}
\mathcal{L}_{u(x)}^{(0)}(f) &= \frac{\partial f}{\partial u} - \mathrm{D}_x\left(\frac{\partial f}{\partial u_x}\right) + \mathrm{D}_x^2\left(\frac{\partial f}{\partial u_{2x}}\right) - \mathrm{D}_x^3\left(\frac{\partial f}{\partial u_{3x}}\right) + \mathrm{D}_x^4\left(\frac{\partial f}{\partial u_{4x}}\right) \\
&= 9c_1 u^2 u_x + 2c_2 u_x u_{2x} + 6c_1 uu_{3x} - \mathrm{D}_x(3c_1 u^3 + 6c_2 u_x^2 + 2c_2 uu_{2x} + 2c_2 u_{4x}) \\
&\quad + \mathrm{D}_x^2(2c_2 uu_x) - \mathrm{D}_x^3(3c_1 u^2) + \mathrm{D}_x^4(2c_2 u_x) \\
&= 9c_1 u^2 u_x + 2c_2 u_x u_{2x} + 6c_1 uu_{3x} - (9c_1 u^2 u_x + 14c_2 u_x u_{2x} + 2c_2 uu_{3x} + 2c_2 u_{5x}) \\
&\quad + (6c_2 u_x u_{2x} + 2c_2 uu_{3x}) - (18c_1 u_x u_{2x} + 6c_1 uu_{3x}) \\
&= -6(3c_1 + c_2)u_x u_{2x}. \tag{51}
\end{aligned}
$$

Note that the terms in $u^2 u_x, uu_{3x}$, and u_{5x} dropped out. Hence, $\mathcal{L}_{u(x)}^{(0)}(f) \equiv 0$ leads to $3c_1 + c_2 = 0$. Substituting $c_1 = \frac{1}{3}, c_2 = -1$ into (42) gives (44) as claimed.

Example: As an example in 2D, we readily verify that $f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$ from (46) is a divergence. Applying (50) to f for each component of $\mathbf{u}(x, y) = (u(x, y), v(x, y))$ separately, we straightforwardly verify that $\mathcal{L}_{u(x,y)}^{(0,0)}(f) \equiv 0$ and $\mathcal{L}_{v(x,y)}^{(0,0)}(f) \equiv 0$.

6.2   Removing Divergences and Divergence-equivalent Terms

It is of paramount importance that densities are free of divergences for they could be moved into the flux $\mathbf{J}$ of the conservation law $\mathrm{D}_t \rho + \mathrm{Div}\, \mathbf{J} = 0$. We show how the Euler operator can be used to remove divergences and divergence-equivalent terms in densities. An algorithm to do so is given in [19]. The following examples illustrate the concept behind the algorithm.

Definition: Two scalar differential functions, $f^{(1)}$ and $f^{(2)}$, are *divergence-equivalent* if and only if they differ by the divergence of some vector $\mathbf{V}$, i.e. $f^{(1)} \sim f^{(2)}$ if and only if $f^{(1)} - f^{(2)} = \mathrm{Div}\, \mathbf{V}$. If a scalar expression is divergence-equivalent to zero then it is a divergence.

Example: Any conserved density of the KdV equation (3) is uniform in rank with respect to $W(u) = 2$ and $W(\partial/\partial x) = 1$. The list of all terms of, say, rank 6 is $\mathcal{R} = \{u^3, u_x^2, uu_{2x}, u_{4x}\}$. Now, terms $f^{(1)} = uu_{2x}$ and $f^{(2)} = -u_x^2$ are divergence-equivalent because $f^{(1)} - f^{(2)} = u_x^2 + uu_{2x} = \mathrm{D}_x(uu_x)$. Using (49), note that $\mathcal{L}_{u(x)}^{(0)}(-u_x^2) = 2u_{2x}$ and $\mathcal{L}_{u(x)}^{(0)}(uu_{2x}) = 2u_{2x}$ are equal (therefore, linearly dependent). So, we discard uu_{2x}. Moreover, $u_{4x} = \mathrm{D}_x(u_{3x})$ is a divergence and, as expected, $\mathcal{L}_{u(x)}^{(0)}(u_{4x}) = 0$. So, we can discard u_{4x}. Hence, $\mathcal{R}$ can be replaced by $\mathcal{S} = \{u^3, u_x^2\}$ which is free of divergences and divergence-equivalent terms.

Example: The list of non-constant terms of rank 6 for the Boussinesq equation (14) is

$$
\mathcal{R} = \{\beta^2 u, \beta u^2, u^3, v^2, u_x v, u_x^2, \beta v_x, uv_x, \beta u_{2x}, uu_{2x}, v_{3x}, u_{4x}\}. \tag{52}
$$

Using (49), for every term t_i in $\mathcal{R}$ we compute $\mathbf{v}_i = \mathcal{L}_{\mathbf{u}(x)}^{(0)}(t_i) = (\mathcal{L}_{u(x)}^{(0)}(t_i), \mathcal{L}_{v(x)}^{(0)}(t_i))$. If $\mathbf{v}_i = (0, 0)$ then t_i is discarded and so is $\mathbf{v}_i$. If $\mathbf{v}_i \neq (0, 0)$ we verify whether or not $\mathbf{v}_i$ is

linearly independent of the non-zero vectors $\mathbf{v}_j$, $j = 1, 2, \cdots, i-1$. If independent, the term t_i is kept, otherwise, t_i is discarded and so is $\mathbf{v}_i$.

Upon application of (49), the first six terms in $\mathcal{R}$ lead to linearly independent vectors $\mathbf{v}_1$ through $\mathbf{v}_6$. Therefore, t_1 through t_6 are kept (and so are the corresponding vectors). For $t_7 = \beta v_x$ we compute $\mathbf{v}_7 = \mathcal{L}_{\mathbf{u}(x)}^{(0)}(\beta v_x) = (0,0)$. So, t_7 is discarded and so is $\mathbf{v}_7$. For $t_8 = u v_x$ we get $\mathbf{v}_8 = \mathcal{L}_{\mathbf{u}(x)}^{(0)}(u v_x) = (v_x, -u_x) = -\mathbf{v}_5$. So, t_8 is discarded and so is $\mathbf{v}_8$.

Proceeding in a similar fashion, t_9, t_{10}, t_{11} and t_{12} are discarded. Thus, $\mathcal{R}$ is replaced by

$$\mathcal{S} = \{\beta^2 u, \beta u^2, u^3, v^2, u_x v, u_x^2\}, \tag{53}$$

which is free of divergences and divergence-equivalent terms.

Example: In the 2D case, $f^{(1)} = (u_x - u_{2x})v_y$ and $f^{(2)} = (u_y - u_{xy})v_x$ are divergence-equivalent since $f^{(1)} - f^{(2)} = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x = \mathrm{Div}\,(u v_y - u_x v_y, -u v_x + u_x v_x)$. Using (50), note that $\mathcal{L}_{\mathbf{u}(x,y)}^{(0)}(f^{(1)}) = \mathcal{L}_{\mathbf{u}(x,y)}^{(0)}(f^{(2)}) = (-v_{xy} - v_{xxy}, -u_{xy} + u_{xxy})$.

6.3 Higher Euler Operators

To compute $\mathbf{F} = \mathrm{Div}^{-1}(f)$ or, in the 1D case, $F = \mathrm{D}_x^{-1}(f) = \int f\,dx$, we need higher-order versions of the variational derivative, called *higher Euler operators* [23, 26] or *Lie-Euler operators* [6]. The general formulas in terms of differential forms are given in [26, p. 367]. We give calculus-based formulas for the 1D and 2D cases (see [19] for the 3D case).

Definition: The *higher Euler operators* in 1D (with variable x)[7] are

$$\mathcal{L}_{\mathbf{u}(x)}^{(i)} = \sum_{k=i}^{\infty} \binom{k}{i} (-\mathrm{D}_x)^{k-i} \frac{\partial}{\partial \mathbf{u}_{kx}}, \tag{54}$$

where $\binom{k}{i}$ is the binomial coefficient. Note that the higher Euler operator for $i = 0$ matches the variational derivative in (49).

Example: The first three higher Euler operators in 1D for component $u(x)$ are

$$\begin{aligned}
\mathcal{L}_{u(x)}^{(1)} &= \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x^3 \frac{\partial}{\partial u_{4x}} + \cdots, \\
\mathcal{L}_{u(x)}^{(2)} &= \frac{\partial}{\partial u_{2x}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3x}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4x}} - 10\mathrm{D}_x^3 \frac{\partial}{\partial u_{5x}} + \cdots, \\
\mathcal{L}_{u(x)}^{(3)} &= \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x \frac{\partial}{\partial u_{4x}} + 10\mathrm{D}_x^2 \frac{\partial}{\partial u_{5x}} - 20\mathrm{D}_x^3 \frac{\partial}{\partial u_{6x}} + \cdots.
\end{aligned} \tag{55}$$

Definition: The *higher Euler operators* in 2D (with variables x, y) are

$$\mathcal{L}_{\mathbf{u}(x,y)}^{(i_x, i_y)} = \sum_{k_x = i_x}^{\infty} \sum_{k_y = i_y}^{\infty} \binom{k_x}{i_x}\binom{k_y}{i_y} (-\mathrm{D}_x)^{k_x - i_x} (-\mathrm{D}_y)^{k_y - i_y} \frac{\partial}{\partial \mathbf{u}_{k_x x\, k_y y}}. \tag{56}$$

---

[7]Variable t in $\mathbf{u}(\mathbf{x}; t)$ is suppressed because it is a parameter in the higher Euler operators.

Note that the higher Euler operator for $i_x = i_y = 0$ matches the variational derivative in (50).

Example: The first higher Euler operators for component $u(x, y)$ are

$$
\begin{aligned}
\mathcal{L}^{(1,0)}_{u(x,y)} &= \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} - \mathrm{D}_y \frac{\partial}{\partial u_{xy}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} + 2\mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{2xy}} - \cdots, \\
\mathcal{L}^{(0,1)}_{u(x,y)} &= \frac{\partial}{\partial u_y} - 2\mathrm{D}_y \frac{\partial}{\partial u_{2y}} - \mathrm{D}_x \frac{\partial}{\partial u_{yx}} + 3\mathrm{D}_y^2 \frac{\partial}{\partial u_{3y}} + 2\mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{x2y}} - \cdots, \\
\mathcal{L}^{(1,1)}_{u(x,y)} &= \frac{\partial}{\partial u_{xy}} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2xy}} - 2\mathrm{D}_y \frac{\partial}{\partial u_{x2y}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3xy}} + 4\mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{2x2y}} + \cdots, \\
\mathcal{L}^{(2,1)}_{u(x,y)} &= \frac{\partial}{\partial u_{2xy}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3xy}} - 2\mathrm{D}_y \frac{\partial}{\partial u_{2x2y}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u_{2x3y}} - \cdots.
\end{aligned}
\tag{57}
$$

The higher Euler operators are useful in their own right as the following theorem [23] in 1D shows.

Theorem: A necessary and sufficient condition for a function f to be an rth order derivative, i.e. there exists a scalar differential function F so that $f = D_x^r F$, is that $\mathcal{L}^{(i)}_{\mathbf{u}(x)}(f) \equiv 0$ for $i = 0, 1, \ldots, r-1$.

6.4 Homotopy Operators

Armed with the higher Euler operators we now turn to the homotopy operator which will allow us to reduce the computation of $\mathbf{F} = \mathrm{Div}^{-1}(f)$, or in the 1D case $F = \mathrm{D}_x^{-1}(f) = \int f \, dx$, to a single integral with respect to an auxiliary variable denoted by λ (not to be confused with λ in Section 4). Amazingly, the homotopy operator circumvents integration by parts (in multi-dimensions involving arbitrary functions) and reduces the inversion of the total divergence operator, Div, to a problem of single-variable calculus.

As mentioned in Section 6.1, Div^{-1} is defined up to a divergence-free (curl) term. In 3D, Div^{-1} is an equivalence class $\mathrm{Div}^{-1}(f) = \mathbf{F} + \boldsymbol{\nabla} \times \mathbf{K}$ where $\mathbf{K}$ is an arbitrary vector differential function. The homotopy operator computes a particular $\mathbf{K}$.

The homotopy operator in terms of differential forms is given in [26, p. 372]. Below we give calculus-based formulas for the homotopy operators in 1D and 2D which are easy to implement in CAS. The explicit formulas in 3D are analogous (see [19]).

Definition: The *homotopy operator* in 1D (with variable x)[8] is

$$
\mathcal{H}_{\mathbf{u}(x)}(f) = \int_0^1 \sum_{j=1}^N I_{u_j}(f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda},
\tag{58}
$$

where u_j is the jth component of $\mathbf{u}$ and the integrand $I_{u_j}(f)$ is given by

$$
I_{u_j}(f) = \sum_{i=0}^\infty \mathrm{D}_x^i \left(u_j \, \mathcal{L}^{(i+1)}_{u_j(x)}(f) \right).
\tag{59}
$$

---

[8]Variable t in $\mathbf{u}(\mathbf{x}; t)$ is suppressed because it is a parameter in the homotopy operators.

The integrand involves the 1D higher Euler operators in (54). In (58), N is the number of dependent variables and $I_{u_j}(f)[\lambda \mathbf{u}]$ means that in $I_{u_j}(f)$ we replace $\mathbf{u}(x) \to \lambda \mathbf{u}(x)$, $\mathbf{u}_x(x) \to \lambda \mathbf{u}_x(x)$, etc.. In practice, we first add the $I_{u_j}(f)$ and then scale the variables. The sum in $I_{u_j}(f)$ terminates at $i = p - 1$ where p is the order of u_j in f.

Given an exact function f, the second question, that is how to compute $F = \mathrm{D}_x^{-1}(f) = \int f\, dx$, is then answered by the following theorem [26, p. 372].

Theorem: For an exact function f, one has $F = \mathcal{H}_{\mathbf{u}(x)}(f)$.

Thus, in the 1D case, applying the homotopy operator (58) allows one to bypass integration by parts. As an experiment, one can start from some function $\tilde{F}$, compute $f = \mathrm{D}_x \tilde{F}$, then compute $F = \mathcal{H}_{\mathbf{u}(x)}(f)$, and finally verify that $F - \tilde{F}$ is a constant.

Example: Using the homotopy operator (58) with (59), we recompute (45). Since f in (42) is of order $p = 4$ in $u_1(x) = u(x)$, the sum in (59) terminates at $i = p - 1 = 3$. Hence,

$$
\begin{aligned}
I_u(f) &= u\mathcal{L}_{u(x)}^{(1)}(f) + \mathrm{D}_x\left(u\mathcal{L}_{u(x)}^{(2)}(f)\right) + \mathrm{D}_x^2\left(u\mathcal{L}_{u(x)}^{(3)}(f)\right) + \mathrm{D}_x^3\left(u\mathcal{L}_{u(x)}^{(4)}(f)\right) \\
&= u\frac{\partial f}{\partial u_x} - 2u\mathrm{D}_x\left(\frac{\partial f}{\partial u_{2x}}\right) + 3u\mathrm{D}_x^2\left(\frac{\partial f}{\partial u_{3x}}\right) - 4u\mathrm{D}_x^3\left(\frac{\partial f}{\partial u_{4x}}\right) + \mathrm{D}_x\left(u\frac{\partial f}{\partial u_{2x}} - 3u\mathrm{D}_x\left(\frac{\partial f}{\partial u_{3x}}\right)\right. \\
&\quad \left. +6u\mathrm{D}_x^2\left(\frac{\partial f}{\partial u_{4x}}\right)\right) + \mathrm{D}_x^2\left(u\frac{\partial f}{\partial u_{3x}} - 4u\mathrm{D}_x\left(\frac{\partial f}{\partial u_{4x}}\right)\right) + \mathrm{D}_x^3\left(u\frac{\partial f}{\partial u_{4x}}\right) \\
&= u^4 + 8u^2 u_{2x} + 6uu_{4x} + 4uu_x^2 - 4\mathrm{D}_x\left(2u^2 u_x + 3uu_{3x}\right) + \mathrm{D}_x^2\left(u^3 + 8uu_{2x}\right) - 2\mathrm{D}_x^3\left(uu_x\right) \\
&= u^4 - 6uu_x^2 + 3u^2 u_{2x} + 2u_{2x}^2 - 4u_x u_{3x}.
\end{aligned}
\tag{60}
$$

Formula (58), with $\mathbf{u} = u_1 = u$, requires an integration with respect to λ :

$$
\begin{aligned}
F = \mathcal{H}_{u(x)}(f) &= \int_0^1 I_u(f)[\lambda u]\, \frac{d\lambda}{\lambda} \\
&= \int_0^1 \left(\lambda^3 u^4 - 6\lambda^2 uu_x^2 + 3\lambda^2 u^2 u_{2x} + 2\lambda u_{2x}^2 - 4\lambda u_x u_{3x}\right) d\lambda \\
&= \frac{1}{4}u^4 - 2uu_x^2 + u^2 u_{2x} + u_{2x}^2 - 2u_x u_{3x}.
\end{aligned}
\tag{61}
$$

The crux of the homotopy operator method is that the integration by parts of a differential expression like (44), which involves an arbitrary function $u(x)$ and its derivatives, can be reduced to a standard integration of a polynomial in λ.

Example: For a system with $N = 2$ components, $\mathbf{u}(x) = (u_1(x), u_2(x)) = (u(x), v(x))$, the homotopy operator formulas are

$$
\mathcal{H}_{\mathbf{u}(x)}(f) = \int_0^1 \left(I_u(f) + I_v(f)\right)[\lambda \mathbf{u}]\, \frac{d\lambda}{\lambda},
\tag{62}
$$

with

$$
I_u(f) = \sum_{i=0}^{\infty} \mathrm{D}_x^i\left(u\, \mathcal{L}_{u(x)}^{(i+1)}(f)\right) \quad \text{and} \quad I_v(f) = \sum_{i=0}^{\infty} \mathrm{D}_x^i\left(v\, \mathcal{L}_{v(x)}^{(i+1)}(f)\right).
\tag{63}
$$

Example: Consider $f = 3u_x v^2 \sin u - u_x^3 \sin u - 6vv_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$, which is no longer polynomial in the two ($N = 2$) dependent variables $u(x)$ and $v(x)$. Applying Euler operator (48) for each component of $\mathbf{u}(x) = (u(x), v(x))$ separately, we quickly verify that $\mathcal{L}_{u(x)}^{(0)}(f) \equiv 0$ and $\mathcal{L}_{v(x)}^{(0)}(f) \equiv 0$. Hence, f is exact. Integration by parts (by hand) gives

$$F = \int f \, dx = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u. \tag{64}$$

Currently, CAS like *Mathematica*, *Maple*[9] and *Reduce* fail this integration due to the presence of trigonometric functions.

We now recompute (64) with the homotopy operator formulas (62) and (63). First,

$$
\begin{aligned}
I_u(f) &= u\mathcal{L}_{u(x)}^{(1)}(f) + \mathrm{D}_x\left(u\mathcal{L}_{u(x)}^{(2)}(f)\right) = u\frac{\partial f}{\partial u_x} - 2u\mathrm{D}_x\left(\frac{\partial f}{\partial u_{2x}}\right) + \mathrm{D}_x\left(u\frac{\partial f}{\partial u_{2x}}\right) \\
&= 3uv^2 \sin u - uu_x^2 \sin u + 2u_x^2 \cos u. \tag{65}
\end{aligned}
$$

Second,

$$
\begin{aligned}
I_v(f) &= v\mathcal{L}_{v(x)}^{(1)}(f) + \mathrm{D}_x\left(v\mathcal{L}_{v(x)}^{(2)}(f)\right) = v\frac{\partial f}{\partial v_x} - 2v\mathrm{D}_x\left(\frac{\partial f}{\partial v_{2x}}\right) + \mathrm{D}_x\left(v\frac{\partial f}{\partial v_{2x}}\right) \\
&= -6v^2 \cos u + 8v_x^2. \tag{66}
\end{aligned}
$$

Finally, using (62),

$$
\begin{aligned}
F = \mathcal{H}_{\mathbf{u}(x)}(f) &= \int_0^1 (I_u(f) + I_v(f))\,[\lambda\mathbf{u}]\,\frac{d\lambda}{\lambda} \\
&= \int_0^1 \left(3\lambda^2 uv^2 \sin(\lambda u) - \lambda^2 uu_x^2 \sin(\lambda u) + 2\lambda u_x^2 \cos(\lambda u) - 6\lambda v^2 \cos(\lambda u) + 8\lambda v_x^2\right) d\lambda \\
&= u_x^2 \cos u - 3v^2 \cos u + 4v_x^2. \tag{67}
\end{aligned}
$$

In contrast to the integration in (64) which involved arbitrary functions $u(x)$ and $v(x)$, in (67) we integrated by parts with respect to the auxiliary variable λ. Any CAS can do that!

We now turn to inverting the Div operator using the homotopy operator.

Definition: We define the *homotopy operator* in 2D (with variables x, y) through its two components $(\mathcal{H}_{\mathbf{u}(x,y)}^{(x)}(f), \mathcal{H}_{\mathbf{u}(x,y)}^{(y)}(f))$. The x-component of the operator is given by

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(x)}(f) = \int_0^1 \sum_{j=1}^{N} I_{u_j}^{(x)}(f)[\lambda\mathbf{u}]\,\frac{d\lambda}{\lambda}, \tag{68}$$

with

$$I_{u_j}^{(x)}(f) = \sum_{i_x=0}^{\infty}\sum_{i_y=0}^{\infty} \left(\frac{1 + i_x}{1 + i_x + i_y}\right) \mathrm{D}_x^{i_x} \mathrm{D}_y^{i_y}\left(u_j\,\mathcal{L}_{u_j(x,y)}^{(1+i_x,i_y)}(f)\right). \tag{69}$$

---

[9]Version 9.5 of *Maple* can integrate such expressions as a result of our interactions with the developers.

Likewise, the y-component is given by

$$\mathcal{H}^{(y)}_{\mathbf{u}(x,y)}(f) = \int_0^1 \sum_{j=1}^N I^{(y)}_{u_j}(f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}, \tag{70}$$

with

$$I^{(y)}_{u_j}(f) = \sum_{i_x=0}^\infty \sum_{i_y=0}^\infty \left(\frac{1 + i_y}{1 + i_x + i_y} \right) \mathrm{D}_x^{i_x} \mathrm{D}_y^{i_y} \left(u_j\, \mathcal{L}^{(i_x,1+i_y)}_{u_j(x,y)}(f) \right). \tag{71}$$

Integrands (69) and (71) involve the 2D higher Euler operators in (56). The question how to compute $\mathbf{F} = (F_1, F_2) = \mathrm{Div}^{-1}(f)$ is then answered by the following theorem.

Theorem: If f is a divergence, then $\mathbf{F} = (F_1, F_2) = \mathrm{Div}^{-1}(f) = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)}(f), \mathcal{H}^{(y)}_{\mathbf{u}(x,y)}(f))$.
The superscripts (x) and (y) remind us which components of $\mathbf{F}$ we are computing. As a matter of testing, we can start from some vector $\tilde{\mathbf{F}}$ and compute $f = \mathrm{Div}\,\tilde{\mathbf{F}}$. Next, we compute $\mathbf{F} = (F_1, F_2) = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)}(f), \mathcal{H}^{(y)}_{\mathbf{u}(x,y)}(f))$ and, finally, verify that $\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F}$ is divergence free.

Example: Using (46), we show how application of the 2D homotopy operator leads to (47), up to a divergence free vector. Consider $f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$, which is the divergence of $\mathbf{F}$ in (47). In order to compute $\mathrm{Div}^{-1}(f)$, we use (69) for the u component in $\mathbf{u}(x,y) = (u(x,y), v(x,y))$:

$$
\begin{aligned}
I^{(x)}_u(f) &= u\mathcal{L}^{(1,0)}_{u(x,y)}(f) + \mathrm{D}_x\left(u\mathcal{L}^{(2,0)}_{u(x,y)}(f) \right) + \frac{1}{2}\mathrm{D}_y\left(u\mathcal{L}^{(1,1)}_{u(x,y)}(f) \right) \\
&= u\left(\frac{\partial f}{\partial u_x} - 2\mathrm{D}_x\frac{\partial f}{\partial u_{2x}} - \mathrm{D}_y\frac{\partial f}{\partial u_{xy}} \right) + \mathrm{D}_x\left(u\frac{\partial f}{\partial u_{2x}} \right) + \frac{1}{2}\mathrm{D}_y\left(u\frac{\partial f}{\partial u_{xy}} \right) \\
&= uv_y + \frac{1}{2}u_y v_x - u_x v_y + \frac{1}{2}uv_{xy}.
\end{aligned} \tag{72}
$$

Similarly, for the v component of $\mathbf{u}(x,y) = (u(x,y), v(x,y))$ we get

$$I^{(x)}_v(f) = v\mathcal{L}^{(1,0)}_{v(x,y)}(f) = v\frac{\partial f}{\partial v_x} = -u_y v + u_{xy} v. \tag{73}$$

Hence, using (68),

$$
\begin{aligned}
F_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)}(f) = \int_0^1 \left(I^{(x)}_u(f) + I^{(x)}_v(f) \right)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \lambda \left(uv_y + \frac{1}{2}u_y v_x - u_x v_y + \frac{1}{2}uv_{xy} - u_y v + u_{xy} v \right) d\lambda \\
&= \frac{1}{2}uv_y + \frac{1}{4}u_y v_x - \frac{1}{2}u_x v_y + \frac{1}{4}uv_{xy} - \frac{1}{2}u_y v + \frac{1}{2}u_{xy} v.
\end{aligned} \tag{74}
$$

Without showing the details, using (70) and (71) we compute analogously

$$F_2 = \mathcal{H}^{(y)}_{\mathbf{u}(x,y)}(f) = \int_0^1 \left(I^{(y)}_u(f) + I^{(y)}_v(f) \right)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

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$$= \int_0^1 \lambda \left(-uv_x - \frac{1}{2}uv_{2x} + \frac{1}{2}u_xv_x + u_xv - u_{2x}v \right) d\lambda$$

$$= -\frac{1}{2}uv_x - \frac{1}{4}uv_{2x} + \frac{1}{4}u_xv_x + \frac{1}{2}u_xv - \frac{1}{2}u_{2x}v. \tag{75}$$

Now we can readily verify that the resulting vector

$$\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}uv_y + \frac{1}{4}u_yv_x - \frac{1}{2}u_xv_y + \frac{1}{4}uv_{xy} - \frac{1}{2}u_yv + \frac{1}{2}u_{xy}v \\ -\frac{1}{2}uv_x - \frac{1}{4}uv_{2x} + \frac{1}{4}u_xv_x + \frac{1}{2}u_xv - \frac{1}{2}u_{2x}v \end{pmatrix} \tag{76}$$

differs from $\tilde{\mathbf{F}} = (uv_y - u_xv_y, -uv_x + u_xv_x)$ by a divergence-free vector

$$\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}uv_y - \frac{1}{4}u_yv_x - \frac{1}{2}u_xv_y - \frac{1}{4}uv_{xy} = \frac{1}{2}u_yv - \frac{1}{2}u_{xy}v \\ -\frac{1}{2}uv_x + \frac{1}{4}uv_{2x} + \frac{3}{4}u_xv_x - \frac{1}{2}u_xv + \frac{1}{2}u_{2x}v \end{pmatrix}. \tag{77}$$

As mentioned in Section 6.1, $\mathbf{K}$ can be written as $(\mathrm{D}_y\phi, -\mathrm{D}_x\phi)$ with $\phi = \frac{1}{2}uv - \frac{1}{4}uv_x - \frac{1}{2}u_xv$.

7 Computation of Conservation Laws

In this section we apply the Euler and homotopy operators to compute density-flux pairs for the four PDEs in Section 3. Using the KdV equation (3) we illustrate the steps for a 1D case (space variable x) and dependent variable $u(x,t)$. Part of the computations for this example were done in the previous sections. The Boussinesq equation (14), also in 1D, is used to illustrate the case with two dependent variables $u(x,t)$ and $v(x,t)$ and an auxiliary parameter with weight.

The LL equation (7), still in 1D, has three dependent variables $u(x,t), v(x,t)$ and $w(x,t)$. The three coupling constants have weights which makes the computations more cumbersome. However, the multi-uniformity of (7) helps to reduce the complexity. The shallow-water wave equations (10) illustrate the computations in 2D (two space variables) and four dependent variables. Again, we can take advantage of the multi-uniformity of (10) to control "expression swell" of the computations.

We use a direct approach to compute conservation laws, $\mathrm{D}_t\,\rho + \mathrm{Div}\,\mathbf{J} = 0$, of polynomial systems of nonlinear PDEs. First, we build the candidate density ρ as a linear combination (with constant coefficients c_i) of terms that are uniform in rank with respect to the scaling symmetry of the PDE. In doing so, we dynamically remove divergences and divergence-equivalent terms to get the shortest possible candidate density.

Second, we evaluate $\mathrm{D}_t\rho$ on solutions of the PDE, thus removing all time derivatives from the problem. The resulting expression, $E = -\mathrm{D}_t\rho$, must be a divergence of the as yet unknown flux. Thus, we compute $\mathcal{L}_{\mathbf{u(x)}}^{(\mathbf{0})}(E)$ and set the coefficients of like terms to zero. This leads to a linear system for the undetermined coefficients c_i. If the given PDE has arbitrary constant parameters, then the linear system is parametrized. Careful analysis of the eliminant is needed to find all solution branches, and, when applicable, the conditions on the parameters. For each branch, the solution of the linear system is substituted into ρ and E.

Third, we use the homotopy operator $\mathcal{H}_{\mathbf{u}(\mathbf{x})}$ to compute $\mathbf{J} = \mathrm{Div}^{-1}(E)$. The computations are done with our *Mathematica* packages [18]. Recall that $\mathbf{J}$ is only defined up a curl term. Removing the curl term in $\mathbf{J}$ may lead to a shorter flux. Inversion of Div via the homotopy operator therefore does not guarantee the shortest flux.

7.1 Conservation Laws for the KdV Equation

In (33) we gave the first three density-flux pairs of (3). As an example, we will compute $\rho^{(3)}$ and $J^{(3)}$. The weights are $W(\partial/\partial x) = 1$ and $W(u) = 2$. Hence, density $\rho^{(3)}$ has rank 6 and flux $J^{(3)}$ has rank 8. The algorithm has three steps:

Step 1: **Construct the form of the density**

Start from $\mathcal{V} = \{u\}$, i.e. the list of dependent variables (and parameters with weight, if applicable). Construct the set $\mathcal{M}$ of all non-constant monomials of (selected) rank 6 or less (without derivatives). Thus, $\mathcal{M} = \{u^3, u^2, u\}$. Next, for each monomial in $\mathcal{M}$, introduce the needed x-derivatives so that each term has rank 6. Since $W(\partial/\partial x) = 1$, use

$$\frac{\partial^2 u^2}{\partial x^2} = (2uu_x)_x = 2u_x^2 + 2uu_{2x}, \qquad \frac{\partial^4 u}{\partial x^4} = u_{4x}. \tag{78}$$

Ignore the highest-order terms (typically the last terms) in each of the right hand sides of (78). Augment $\mathcal{M}$ with the remaining terms (deleting numerical factors) to get $\mathcal{R} = \{u^3, u_x^2\}$.

Here $\mathcal{S} = \mathcal{R}$ since $\mathcal{R}$ is free of divergences and divergent-equivalent terms. Linearly combine the terms in $\mathcal{S}$ with constant coefficients to get the candidate density:

$$\rho = c_1 u^3 + c_2 u_x^2, \tag{79}$$

which is (27).

Step 2: **Compute the undetermined constants c_i**

Compute $\mathrm{D}_t \rho$ and use (3) to eliminate u_t and u_{tx}. As shown in (28), this gives

$$E = -\mathrm{D}_t \rho = 3c_1 u^3 u_x + 3c_1 u^2 u_{3x} + 2c_2 u_x^3 + 2c_2 u u_x u_{2x} + 2c_2 u_x u_{4x}. \tag{80}$$

Since $E = -\mathrm{D}_t \rho = \mathrm{D}_x J$, the expression E must be exact. As shown in (51), $\mathcal{L}_{u(x)}^{(0)}(E) = -6(3c_1 + c_2)u_x u_{2x} \equiv 0$ leads to $c_1 = \frac{1}{3}, c_2 = -1$ and upon substitution into (79) to

$$\rho = \frac{1}{3}u^3 - u_x^2 \tag{81}$$

which is $\rho^{(3)}$ in (33).

Step 3: **Compute the flux J**

Substitute $c_1 = \frac{1}{3}, c_2 = -1$ into (80) to get

$$E = u^3 u_x - 2u_x^3 - 2uu_x u_{2x} + u^2 u_{3x} - 2u_x u_{4x}. \tag{82}$$

As shown in (61), application of (58) with (59) to (82) gives

$$J = \frac{1}{4}u^4 - 2uu_x^2 + u^2 u_{2x} + u_{2x}^2 - 2u_x u_{3x}, \tag{83}$$

which is $J^{(3)}$ in (33).

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7.2 Conservation Laws for the Boussinesq Equation

The first few density-flux pairs of (14) were given in (35). Equation (14) has weights $W(\frac{\partial}{\partial x}) = 1$, $W(u) = W(\beta) = 2$, and $W(v) = 3$. We show the computation of $\rho^{(4)}$ and $J^{(4)}$ of ranks 6 and 7, respectively. The presence of the auxiliary parameter β with weight complicates matters.

Step 1: **Construct the form of the density**

Augment the list of dependent variables with the parameter β (with non-zero weight). Hence, $\mathcal{V} = \{u, v, \beta\}$. Construct $\mathcal{M} = \{\beta^2 u, \beta u^2, \beta u, \beta v, u^3, u^2, u, v^2, v, uv\}$, which contains all non-constant monomials of (chosen) rank 6 or less (without derivatives). Next, for each term in $\mathcal{M}$, introduce the right number of x-derivatives so that each term has rank 6. For example,

$$\frac{\partial^2 \beta u}{\partial x^2} = \beta u_{2x}, \quad \frac{\partial^2 u^2}{\partial x^2} = 2u_x^2 + 2uu_{2x}, \quad \frac{\partial^4 u}{\partial x^4} = u_{4x}, \quad \frac{\partial(uv)}{\partial x} = u_x v + uv_x, \quad etc.. \quad (84)$$

Ignore the highest-order terms (typically the last terms) in each of the right hand sides of (84). Augment $\mathcal{M}$ with the remaining terms, after deleting numerical factors, to get $\mathcal{R} = \{\beta^2 u, \beta u^2, u^3, v^2, u_x v, u_x^2, \beta v_x, uv_x, \beta u_{2x}, uu_{2x}, v_{3x}, u_{4x}\}$ as in (52). As shown in (53), removal of divergences and divergence-free terms in $\mathcal{R}$ leads to $\mathcal{S} = \{\beta^2 u, \beta u^2, u^3, v^2, u_x v, u_x^2\}$. Linearly combine the terms in $\mathcal{S}$ to get

$$\rho = c_1 \beta^2 u + c_2 \beta u^2 + c_3 u^3 + c_4 v^2 + c_5 u_x v + c_6 u_x^2. \quad (85)$$

Step 2: **Compute the undetermined constants** c_i

Compute $E = -D_t \rho$ and use (14) to eliminate u_t, u_{tx}, and v_t. As shown in (32), this gives

$$E = (c_1 \beta^2 + 2c_2 \beta u + 3c_3 u^2)v_x + (c_5 v + 2c_6 u_x)v_{2x} + (2c_4 v + c_5 u_x)(\beta u_x - 3uu_x - \alpha u_{3x}). \quad (86)$$

$E = -D_t \rho = D_x J$ must be exact. Thus, apply (48) and require that $\mathcal{L}_{u(x)}^{(0)}(E) = \mathcal{L}_{v(x)}^{(0)}(E) \equiv 0$. Group like terms. Set their coefficients equal to zero to obtain the parametrized system

$$\beta(c_2 - c_4) = 0, \quad c_3 + c_4 = 0, \quad c_5 = 0, \quad \alpha c_5 = 0, \quad \beta c_5 = 0, \quad \alpha c_4 - c_6 = 0. \quad (87)$$

Investigate the eliminant of the system. Set $c_1 = 1$, to obtain the solution

$$c_1 = 1, \quad c_2 = c_4, \quad c_3 = -c_4, \quad c_5 = 0, \quad c_6 = \alpha c_4, \quad (88)$$

which is valid irrespective of the values of α and β. Substitute (88) into (85) to get

$$\rho = \beta^2 u + c_4(\beta u^2 - u^3 + v^2 + \alpha u_x^2). \quad (89)$$

The density must be split into independent pieces. Indeed, since c_4 is arbitrary, set $c_4 = 0$ or $c_4 = 1$, thus splitting (89) into two independent densities, $\rho = \beta^2 u$ and

$$\rho = \beta u^2 - u^3 + v^2 + \alpha u_x^2, \quad (90)$$

which are $\rho^{(1)}$ and $\rho^{(4)}$ in (35).

Step 3: **Compute the flux J**

Compute the flux corresponding to ρ in (90). Substitute (88) into (86). Take the terms in c_4 and set $c_4 = 1$. Thus,

$$E = 2\beta u v_x + 2\beta u_x v - 3u^2 v_x - 6u u_x v + 2\alpha u_x v_{2x} - 2\alpha u_{3x} v. \tag{91}$$

Apply (62) and (63) to (91). For E of order 3 in $u(x)$, compute

$$
\begin{aligned}
I_u(E) &= u\mathcal{L}_{u(x)}^{(1)}(E) + \mathrm{D}_x\left(u\mathcal{L}_{u(x)}^{(2)}(E)\right) + \mathrm{D}_x^2\left(u\mathcal{L}_{u(x)}^{(3)}(E)\right) \\
&= u\frac{\partial E}{\partial u_x} - 2u\mathrm{D}_x\left(\frac{\partial E}{\partial u_{2x}}\right) + 3u\mathrm{D}_x^2\left(\frac{\partial E}{\partial u_{3x}}\right) + \mathrm{D}_x\left(u\frac{\partial E}{\partial u_{2x}} - 3u\mathrm{D}_x\left(\frac{\partial E}{\partial u_{3x}}\right)\right) + \mathrm{D}_x^2\left(u\frac{\partial E}{\partial u_{3x}}\right) \\
&= 2\beta uv - 6u^2 v - 4\alpha u v_{2x} + 6\alpha \mathrm{D}_x\left(uv_x\right) - 2\alpha \mathrm{D}_x^2\left(uv\right) \\
&= 2\beta uv - 6u^2 v - 2\alpha u_{2x} v + 2\alpha u_x v_x. \tag{92}
\end{aligned}
$$

With E is of order 2 in $v(x)$, subsequently compute

$$
\begin{aligned}
I_v(E) &= v\mathcal{L}_{v(x)}^{(1)}(E) + \mathrm{D}_x\left(v\mathcal{L}_{v(x)}^{(2)}(E)\right) = v\frac{\partial E}{\partial v_x} - 2v\mathrm{D}_x\left(\frac{\partial E}{\partial v_{2x}}\right) + \mathrm{D}_x\left(v\frac{\partial E}{\partial v_{2x}}\right) \\
&= 2\beta uv - 3u^2 v - 4\alpha uv_{2x} + 2\alpha \mathrm{D}_x\left(u_x v\right) = 2\beta uv - 3u^2 v - 2\alpha u_{2x} v + 2\alpha u_x v_x. \tag{93}
\end{aligned}
$$

Formula (62) with $\mathbf{u}(x) = (u(x), v(x))$ requires an integration with respect to λ. Hence,

$$
\begin{aligned}
J &= \mathcal{H}_{u(x)}(E) = \int_0^1 (I_u(E) + I_v(E))[\lambda \mathbf{u}]\frac{d\lambda}{\lambda} \\
&= \int_0^1 \left(4\beta \lambda uv - 9\lambda^2 u^2 v - 4\alpha \lambda u_{2x} v + 4\alpha \lambda u_x v_x\right) d\lambda \\
&= 2\beta uv - 3u^2 v - 2\alpha u_{2x} v + 2\alpha u_x v_x, \tag{94}
\end{aligned}
$$

which is $J^{(4)}$ in (35). Set $\beta = 1$ at the end of the computations.

7.3 Conservation Laws for the Landau-Lifshitz Equation

Without showing full details, we will compute the six density-flux pairs (36) for (7). The weights for (7) are $W(u) = W(v) = W(w) = a$, $W(\alpha) = W(\beta) = W(\gamma) = 2$, $W(\frac{\partial}{\partial t}) = a + 2$, where a is an arbitrary non-negative integer or rational number. This example involves three constants ($\alpha, \beta,$ and γ) with weights which makes the computations quite cumbersome unless we take advantage of the multi-uniformity and the cyclic nature of (7).

Step 1: **Construct the form of the density**

Select a small value for a, for example $a = \frac{1}{4}$, and compute a density of, say, rank $\frac{1}{4}$. Start from $\mathcal{V} = \{u, v, w, \alpha, \beta, \gamma\}$, i.e. the list of dependent variables and the parameters with weight. Construct the set $\mathcal{M}$ of all non-constant monomials of the (selected) rank $\frac{1}{4}$ or less (without derivatives). Thus, $\mathcal{M} = \{u, v, w\}$. Next, for each of the monomials in $\mathcal{M}$, introduce

the appropriate number of x-derivatives so that each term has rank $\frac{1}{4}$. No x-derivatives are needed and the removal of divergences or divergence-equivalent terms is irrelevant. Linearly combine the terms in $\mathcal{R}=\mathcal{S}=\{u,v,w\}$ to get a candidate density:

$$\rho = c_1 u + c_2 v + c_3 w. \tag{95}$$

It suffices to continue with $\rho = c_1 u$. The remaining terms follow by cyclic permutation.

Step 2: **Compute the undetermined constants** c_i

Compute $E = -\mathrm{D}_t \rho = -\mathrm{D}_t(c_1 u) = -c_1 u_t$. Use (7) to eliminate u_t. Hence,

$$E = -c_1\Big(v w_{2x} - w v_{2x} + (\gamma - \beta) v w\Big), \tag{96}$$

which is the opposite of the right hand side of the first equation in (7). Since $E = -\mathrm{D}_t\,\rho = \mathrm{D}_x J$, the expression E must be exact. Obviously $\mathcal{L}_{u(x)}^{(0)}(E) \equiv 0$. Compute

$$\mathcal{L}_{v(x)}^{(0)}(E) = \frac{\partial f}{\partial v}(E) - \mathrm{D}_x \frac{\partial f}{\partial v_x}(E) + \mathrm{D}_x^2 \frac{\partial f}{\partial v_{2x}}(E) = -c_1(\gamma - \beta)w,$$

$$\mathcal{L}_{w(x)}^{(0)}(E) = \frac{\partial f}{\partial w}(E) - \mathrm{D}_x \frac{\partial f}{\partial w_x}(E) + \mathrm{D}_x^2 \frac{\partial f}{\partial w_{2x}}(E) = -c_1(\gamma - \beta)v. \tag{97}$$

Set the latter expressions identically equal to zero. Solve $c_1(\beta - \gamma) = 0$ for $c_1 \neq 0$. Set $c_1 = 1$ to get $\rho = u$, subject to the condition $\beta = \gamma$, which confirms the result in (36).

Step 3: **Compute the flux** J

Substitute $c_1 = 1$ and $\beta = \gamma$ into (96) to get $E = w v_{2x} - v w_{2x}$. Apply the homotopy operator (58) with (59) to E. In this example $\mathbf{u}(x) = (u(x), v(x), w(x))$. Obviously, $I_u(E) = 0$, since there is no explicit occurrence of u. Compute

$$I_v(E) = v\mathcal{L}_{v(x)}^{(1)}(E) + \mathrm{D}_x\left(v\mathcal{L}_{v(x)}^{(2)}(E)\right) = v\frac{\partial E}{\partial v_x} - 2v\mathrm{D}_x\left(\frac{\partial E}{\partial v_{2x}}\right) + \mathrm{D}_x\left(v\frac{\partial E}{\partial v_{2x}}\right) = v_x w - v w_x, \tag{98}$$

and, analogously, $I_w(E) = v_x w - v w_x$. Finally, compute

$$\begin{aligned}
J &= \mathcal{H}_{u(x)}(E) = \int_0^1 (I_u(E) + I_v(E) + I_w(E))[\lambda \mathbf{u}]\,\frac{d\lambda}{\lambda} \\
&= \int_0^1 2\lambda\,(v_x w - v w_x)\,d\lambda = v_x w - v w_x,
\end{aligned} \tag{99}$$

which is $J^{(1)}$ in (36).

To compute density $\rho^{(4)}$ which is quadratic in u, v, and w, we would start from rank $\frac{1}{2}$ and repeat the three steps. Step 1 would lead to $\rho = c_1 u^2 + c_2 uv + c_3 v^2 + c_4 uw + c_5 vw + c_6 w^2$. Steps 2 and 3 would result in $c_1 = c_3 = c_6 = 1$. So, $\rho = u^2 + v^2 + w^2$ and $J = 0$, which agrees with $\rho^{(4)}$ and $J^{(4)} = 0$ in (36).

Starting with rank 1, Step 1 would generate a polynomial density with the 15 terms that are homogeneous of degree 4. Steps 2 and 3 would result in $\rho^{(5)}$ and $J^{(5)} = 0$ in (36).

The computation of $\rho^{(6)}$ and $J^{(6)}$ is cumbersome and long. We only indicate the strategy and give partial results. Notice that $\rho^{(6)}$ would have rank $\frac{5}{2}$ if $a = \frac{1}{4}$ and the candidate density would have, amongst others, all homogeneous terms of degree nine, which is undesirable. It turns out that $a = 1$ is a better choice to compute $\rho^{(6)}$, which then has rank 4.

Start from $\mathcal{V}$ as above. Construct the set $\mathcal{M} = \{u^4, u^3, u^2, u, v^4, v^3, \cdots, u^3v, u^3w, \cdots, u^2v^2, u^2w^2, \cdots, u^2vw, v^2uw, \cdots, \alpha u^2, \beta u^2, \cdots, \alpha uv, \cdots, \gamma vw, \cdots, uvw, uv, \cdots, \gamma w\}$ of all non-constant monomials of rank 4 or less (without derivatives). Next, for each of the 61 monomials in $\mathcal{M}$, introduce the needed x-derivatives to make each term rank 4. Use, for example,

$$\frac{\partial^2 u^2}{\partial x^2} = 2u_x^2 + 2uu_{2x}, \quad \frac{\partial u^3}{\partial x} = 3u^2 u_x, \quad \frac{\partial^4 u}{\partial x^4} = u_{4x}, \quad \frac{\partial u^2 v}{\partial x} = 2uu_x v + u^2 v_x, \quad \frac{\partial \alpha u}{\partial x} = \alpha u_x. \quad (100)$$

Ignore the highest-order terms (typically the last terms). Augment $\mathcal{M}$ with the remaining terms (deleting numerical factors). Next, remove all divergences and divergent-equivalent terms to get $\mathcal{S}$ with 47 terms (not shown). Linearly combine the terms in $\mathcal{S}$ to get the candidate density:

$$\begin{aligned}
\rho =\; & c_1 \alpha u^2 + c_2 \beta u^2 + c_3 \gamma u^2 + c_4 u^4 + c_5 \alpha uv + c_6 \beta uv + c_7 \gamma uv + c_8 u^3 v + \cdots + c_{12} u^2 v^2 \\
& + c_{13} uv^3 + \cdots + c_{22} u^2 vw + c_{23} uv^2 w + \cdots + c_{25} \alpha w^3 + c_{26} \beta w^3 + c_{27} \gamma w^3 + \cdots + c_{29} uvw^2 \\
& + c_{30} v^2 w^2 + \cdots + c_{33} w^4 + c_{34} uu_x v + c_{35} u_x v^2 + c_{36} uu_x w + c_{37} u_x vw + c_{38} u_x w^2 + c_{39} u_x^2 \\
& + c_{40} uv_x w + c_{41} vv_x w + c_{42} v_x w^2 + c_{43} u_x v_x + c_{44} v_x^2 + c_{45} u_x w_x + c_{46} v_x w_x + c_{47} w_x^2, \quad (101)
\end{aligned}$$

where only the most indicative terms are explicitly shown.

Compute $D_t \rho$, use (7) to eliminate $u_t, v_t, w_t, u_{tx}, v_{tx}$, and w_{tx}, and require that $\mathcal{L}_{u(x)}^{(0)}(E) = \mathcal{L}_{v(x)}^{(0)}(E) = \mathcal{L}_{w(x)}^{(0)}(E) \equiv 0$ which leads to a linear system of 121 equations (not shown) for c_1 through c_{47}. Substitute the solution (not shown) into (101) to obtain

$$\begin{aligned}
\rho =\; & (\alpha c_{25} + \beta c_{26} + \gamma c_{27})(u^2 + v^2 + w^2) + \frac{1}{2} c_{30} (u^2 + v^2 + w^2)^2 \\
& + c_{47} \left(u_x^2 + v_x^2 + w_x^2 + (\gamma - \alpha)u^2 + (\gamma - \beta)v^2 \right), \quad (102)
\end{aligned}$$

where $c_{25}, c_{26}, c_{27}, c_{30}$, and c_{47} are arbitrary. Split the density into independent pieces:

$$\begin{aligned}
\rho &= u^2 + v^2 + w^2, \quad \rho = (u^2 + v^2 + w^2)^2, \\
\rho &= u_x^2 + v_x^2 + w_x^2 + (\gamma - \alpha)u^2 + (\gamma - \beta)v^2, \quad (103)
\end{aligned}$$

which are $\rho^{(4)}, \rho^{(5)}$, and $\rho^{(6)}$ in (36). Use the homotopy operator to compute the flux corresponding to (102):

$$\begin{aligned}
J =\; & 2c_{47} \Big((vw_x - wv_x)u_{2x} + (wu_x - uw_x)v_{2x} + (uv_x - vu_x)w_{2x} \\
& + (\beta - \gamma)vwu_x + (\gamma - \alpha)uwv_x + (\alpha - \beta)uvw_x \Big). \quad (104)
\end{aligned}$$

$J^{(4)} = J^{(5)} = 0$ since the terms in c_{25}, c_{26}, c_{27}, and c_{30} all dropped out. Set $c_{47} = 1$ to get $J^{(6)}$ in (36).

7.4 Conservation Laws for the Shallow Water Wave Equations

The SWW equations (10) admit weights (19) in which $W(h)$ and $W(\Omega)$ are free. The fact that (10) is multi-uniform is advantageous. Indeed, we can construct a candidate ρ which is uniform in rank for one set of weights and, subsequently, use other choices of weights to split ρ into smaller pieces. This "divide and conquer" strategy drastically reduces the complexity of the computations as was shown in [19].

The first few densities and fluxes were given in (41). We compute $\rho^{(5)}$ and $J^{(5)}$. Note that $\rho^{(5)}$ has rank 3 if we select $W(h) = a = 1$ and $W(\Omega) = b = 2$. However, $\rho^{(5)}$ has rank 4 if we take $W(h) = a = 0$ and $W(\Omega) = b = 2$. If we set $W(h) = 0$, and $W(\Omega) = 3$ then $\rho^{(5)}$ has rank 7. So, the trick is to construct a candidate density which is scaling homogeneous for a *particular* (fixed) choice of a and b in (20) and split the density based on other choices of a and b.

Step 1: **Construct the form of the density**
Start from $\mathcal{V} = \{u, v, \theta, h, \Omega\}$, i.e. the list of variables and parameters with weights. Use (20) with $a = 1, b = 2$, to get $\mathcal{M} = \{\Omega u, \Omega v, \dots, u^3, v^3, \dots, u^2 v, uv^2, \dots, u^2, v^2, \dots, u, v, \theta, h\}$ which has 38 monomials of (chosen) rank 3 or less (without derivatives).

All terms of rank 3 in $\mathcal{M}$ remain untouched. To adjust the rank, differentiate each monomial of rank 2 in $\mathcal{M}$ with respect to x ignoring the highest-order term. For example, in $\frac{du^2}{dx} = 2uu_x$, the term can be ignored since it is a total derivative. The terms $u_x v$ and $-uv_x$ are divergence-equivalent since $\frac{d(uv)}{dx} = u_x v + uv_x$. Keep $u_x v$. Likewise, differentiate each monomial of rank 2 in $\mathcal{M}$ with respect to y and ignore the highest-order term.

Produce the remaining terms for rank 3 by differentiating the monomials of rank 1 in $\mathcal{M}$ with respect to x twice, or y twice, or once with respect to x and y. Again ignore the highest-order terms. Augment the set $\mathcal{M}$ with the derivative terms of rank 3 to get $\mathcal{R} = \{\Omega u, \Omega v, \cdots, uv^2, u_x v, u_x \theta, u_x h, \cdots, u_y v, u_y \theta, \cdots, \theta_y h\}$ which has 36 terms.

Use the "divide and conquer" strategy [19] to select from $\mathcal{R}$ the terms which are of ranks 4 and 7 using the choices $a = 0, b = 2$ and $a = 0, b = 3$, respectively. This gives $\mathcal{S} = \{\Omega\theta, u_x\theta, u_y\theta, v_x\theta, v_y\theta\}$, which happens to be free of divergences and divergence-equivalent terms. So, no further reduction is needed. Linearly combine the terms in $\mathcal{S}$ to get

$$\rho = c_1 \Omega\theta + c_2 u_x\theta + c_3 u_y\theta + c_4 v_x\theta + c_5 v_y\theta. \tag{105}$$

Step 2: **Compute the undetermined constants c_i**
Compute $E = -\mathrm{D}_t\rho$ and use (10) to remove all time derivatives:

$$
\begin{aligned}
E &= -\Big(\frac{\partial\rho}{\partial u_x}u_{tx} + \frac{\partial\rho}{\partial u_y}u_{ty} + \frac{\partial\rho}{\partial v_x}v_{tx} + \frac{\partial\rho}{\partial v_y}v_{ty} + \frac{\partial\rho}{\partial\theta}\theta_t\Big) \\
&= c_2\theta(uu_x + vu_y - 2\Omega v + \tfrac{1}{2}h\theta_x + \theta h_x)_x + c_3\theta(uu_x + vu_y - 2\Omega v + \tfrac{1}{2}h\theta_x + \theta h_x)_y \\
&\quad + c_4\theta(uv_x + vv_y + 2\Omega u + \tfrac{1}{2}h\theta_y + \theta h_y)_x + c_5\theta(uv_x + vv_y + 2\Omega u + \tfrac{1}{2}h\theta_y + \theta h_y)_y \\
&\quad + (c_1\Omega + c_2 u_x + c_3 u_y + c_4 v_x + c_5 v_y)(u\theta_x + v\theta_y).
\end{aligned}
\tag{106}
$$

Require that $\mathcal{L}_{u(x,y)}^{(0,0)}(E) = \mathcal{L}_{v(x,y)}^{(0,0)}(E) = \mathcal{L}_{\theta(x,y)}^{(0,0)}(E) = \mathcal{L}_{h(x,y)}^{(0,0)}(E) \equiv 0$, where, for example, $\mathcal{L}_{u(x,y)}^{(0,0)}$ is given in (50). Gather like terms. Equate their coefficients to zero to obtain

$$c_1 + 2c_3 = 0, \quad c_2 = c_5 = 0, \quad c_1 - 2c_4 = 0, \quad c_3 + c_4 = 0. \tag{107}$$

Set $c_1 = 2$. Substitute the solution

$$c_1 = 2, \ c_2 = 0, \ c_3 = -1, \ c_4 = 1, \ c_5 = 0. \tag{108}$$

into ρ to obtain

$$\rho = 2\Omega\theta - u_y\theta + v_x\theta, \tag{109}$$

which is $\rho^{(5)}$ in (41).

Step 3: **Compute the flux J**

Compute the flux corresponding to (109). To do so, substitute (108) into (106) to obtain

$$\begin{aligned}
E \ = \ & \theta(u_x v_x - u_y v_y + u v_{2x} - u_{2y} v - u_x u_y + v_x v_y - u u_{xy} + v v_{xy} \\
& + 2\Omega u_x + 2\Omega v_y + \tfrac{1}{2}\theta_x h_y - \tfrac{1}{2}\theta_y h_x) + 2\Omega u\theta_x + 2\Omega v\theta_y - u u_y\theta_x \\
& - u_y v\theta_y + u v_x\theta_x + v v_x\theta_y.
\end{aligned} \tag{110}$$

Apply the 2D homotopy formulas in (68)-(71) to $E = \mathrm{Div}\,\mathbf{J} = \mathrm{D}_x J_1 + \mathrm{D}_y J_2$. So, compute

$$\begin{aligned}
I_u^{(x)}(E) \ = \ & u\mathcal{L}_{u(x,y)}^{(1,0)}(E) + \mathrm{D}_x\left(u\mathcal{L}_{u(x,y)}^{(2,0)}(E)\right) + \frac{1}{2}\mathrm{D}_y\left(u\mathcal{L}_{u(x,y)}^{(1,1)}(E)\right) \\
= \ & u\left(\frac{\partial E}{\partial u_x} - 2\mathrm{D}_x\left(\frac{\partial E}{\partial u_{2x}}\right) - \mathrm{D}_y\left(\frac{\partial E}{\partial u_{xy}}\right)\right) + \mathrm{D}_x\left(u\frac{\partial E}{\partial u_{2x}}\right) + \frac{1}{2}\mathrm{D}_y\left(u\frac{\partial E}{\partial u_{xy}}\right) \\
= \ & u v_x\theta + 2\Omega u\theta + \frac{1}{2}u^2\theta_y - u u_y\theta.
\end{aligned} \tag{111}$$

Similarly, compute

$$I_v^{(x)}(E) \ = \ v v_y\theta + \frac{1}{2}v^2\theta_y + u v_x\theta, \tag{112}$$

$$I_\theta^{(x)}(E) \ = \ \frac{1}{2}\theta^2 h_y + 2\Omega u\theta - u u_y\theta + u v_x\theta, \tag{113}$$

$$I_h^{(x)}(E) \ = \ -\frac{1}{2}\theta\theta_y h. \tag{114}$$

Next, compute

$$\begin{aligned}
J_1(\mathbf{u}) \ = \ & \mathcal{H}_{\mathbf{u}(x,y)}^{(x)}(E) \\
= \ & \int_0^1 \left(I_u^{(x)}(E) + I_v^{(x)}(E) + I_\theta^{(x)}(E) + I_h^{(x)}(E)\right)[\lambda\mathbf{u}]\,\frac{d\lambda}{\lambda} \\
= \ & \int_0^1 \left(4\lambda\Omega u\theta + \lambda^2\left(3u v_x\theta + \frac{1}{2}u^2\theta_y - 2u u_y\theta + v v_y\theta + \frac{1}{2}v^2\theta_y \right.\right. \\
& \left.\left. + \frac{1}{2}\theta^2 h_y - \frac{1}{2}\theta\theta_y h\right)\right)d\lambda \\
= \ & 2\Omega u\theta - \frac{2}{3}u u_y\theta + u v_x\theta + \frac{1}{3}v v_y\theta + \frac{1}{6}u^2\theta_y + \frac{1}{6}v^2\theta_y - \frac{1}{6}h\theta\theta_y + \frac{1}{6}h_y\theta^2. \quad (115)
\end{aligned}$$

Likewise, compute $I_u^{(y)}(E), I_v^{(y)}(E), I_\theta^{(y)}(E)$, and $I_h^{(y)}(E)$. Finally, compute

$$
\begin{aligned}
J_2(\mathbf{u}) &= \mathcal{H}_{\mathbf{u}(x,y)}^{(y)}(E) \\
&= \int_0^1 \left(I_u^{(y)}(E) + I_v^{(y)}(E) + I_\theta^{(y)}(E) + I_h^{(y)}(E) \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= 2\Omega v\theta + \frac{2}{3}vv_x\theta - vu_y\theta - \frac{1}{3}uu_x\theta - \frac{1}{6}u^2\theta_x - \frac{1}{6}v^2\theta_x + \frac{1}{6}h\theta\theta_x - \frac{1}{6}h_x\theta^2. \quad (116)
\end{aligned}
$$

Hence,

$$
\mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}, \quad (117)
$$

which matches $\mathbf{J}^{(5)}$ in (41).

8   Conclusions

The variational derivative (zeroth Euler operator) provides a straightforward way to test exactness which is key in the computation of densities. The continuous homotopy operator is a powerful, algorithmic tool to compute fluxes explicitly. Indeed, the homotopy operator allows one to invert the total divergence operator by computing higher variational derivatives followed by a one-dimensional integration with respect to a single auxiliary parameter. The homotopy operator is a universal tool that can be applied to problems in which integration by parts (of arbitrary functions) in multi-variables is crucial.

To reach a wider audience, we intentionally did not use differential forms and the abstract framework of variational bi-complexes and homological algebra. Instead, we extracted the Euler and homotopy operators from their abstract setting and presented them in the language of standard calculus, thereby making them widely applicable to computational problems in the sciences. Our calculus-based formulas can be readily implemented in CAS.

Based on the concept of scaling invariance and using tools of the calculus of variations, we presented a three-step algorithm to symbolically compute polynomial conserved densities and fluxes of nonlinear polynomial systems of PDEs in multi-spatial dimensions. The steps are straightforward: build candidate densities as linear combinations (with undetermined constant coefficients) of terms that are homogeneous with respect to the scaling symmetry of the PDE. Subsequently, use the variational derivative to compute the coefficients, and, finally, use the homotopy operator to compute the flux.

With our method one can search for conservation laws in chemistry, physics, and engineering. Symbolic packages are well suited to assist in the search which covers many fields of application including fluid mechanics, plasma physics, electro-dynamics, gas dynamics, elasticity, nonlinear optics and acoustics, electrical networks, chemical reactions, etc..

Acknowledgements and Dedication

References

[1] Ablowitz, M. J.; Clarkson, P. A. Solitons, Nonlinear Evolution Equations and Inverse Scattering; Cambridge University Press: Cambridge, 1991.

[2] Ablowitz, M. J.; Segur, H. Solitons and the Inverse Scattering Transform; SIAM: Philadelphia, 1981.

[3] Adams, P. J. Symbolic Computation of Conserved Densities and Fluxes for Systems of Partial Differential Equations with Transcendental Nonlinearities; MS Thesis, Dept. Math. & Comp. Scs.; Colorado School of Mines: Golden, Colorado, 2003.

[4] Anco, S. C. J Phys A: Math Gen 2003, 36, 8623-8638.

[5] Anco, S. C.; Bluman, G. Euro J Appl Math 2002, 13, 545-566 & 567-585.

[6] Anderson, I. M. The Variational Bicomplex; Dept. Math., Utah State University: Logan, Utah, November 2004, 318 pages;
http://www.math.usu.edu/~fg_mp/Publications/VB/vb.pdf .

[7] Anderson, I. M. The Vessiot Package, Dept. Math., Utah State University: Logan, Utah, November 2004, 253 pages;
http://www.math.usu.edu/~fg_mp/Pages/SymbolicsPage/VessiotDownloads.html

[8] Barnett, M. P.; Capitani, J. F.; von zur Gathen J.; Gerhard, J. Int J Quan Chem 2004, 100, 80-104.

[9] Cantarella, J.; DeTurck, D.; Gluck, H. Am Math Monthly 2002, 109, 409-442.

[10] Deconinck, B.; Nivala, M. A Maple Package for the Computation of Conservation Laws; http://www.amath.washington.edu/~bernard/papers.html.

[11] Dellar, P. Phys Fluids 2003, 15, 292-297.

[12] Eklund, H. Symbolic Computation of Conserved Densities and Fluxes for Nonlinear Systems of Differential-Difference Equations; MS Thesis, Dept. Math. & Comp. Scs.; Colorado School of Mines: Golden, Colorado, 2003.

[13] Faddeev, L. D.; Takhtajan, L. A. Hamiltonian Methods in the Theory of Solitons; Springer Verlag: Berlin, 1987.

[14] Gao, M.; Kato, Y.; Ito, M. Comp Phys Comm 2002, 148, 242-255.

[15] Gao, M.; Kato, Y.; Ito, M. Comp Phys Comm 2004, 160, 69-89.

[16] Göktaş, Ü.; Hereman, W. J Symb Comp 1997, 24, 591-621.

[17] Göktaş, Ü.; Hereman, W. Phys D 1998, 132, 425-436.

[18] Hereman, W. Mathematica Packages for the Symbolic Computation of Conservation Laws of Partial Differential Equations and Differential-difference Equations; http://www.mines.edu/fs_home/whereman/.

[19] Hereman, W.; Colagrosso, M.; Sayers, R.; Ringler, A.; Deconinck, B.; Nivala, M.; Hickman, M. S. In Differential Equations with Symbolic Computation; Wang, D.; Zheng, Z., Eds.; Birkhäuser Verlag: Basel, 2005.

[20] Hereman, W.; Sanders, J. A.; Sayers, J.; Wang, J. P. In Group Theory and Numerical Methods, CRM Proc. Lect. Ser. **39**, Winternitz, P.; Gomez-Ullate, D., Eds.; AMS: Providence, Rhode Island, 2005, pp. 267-282.

[21] Hickman, M.; Hereman, W. Proc Roy Soc Lond A 2003, 459, 2705-2729.

[22] Krasil'shchik, I. S.; Vinogradov, A. M. Symmetries and Conservation Laws for Differential Equations of Mathematical Physics. AMS: Providence, Rhode Island, 1998.

[23] Kruskal, M. D.; Miura, R. M.; Gardner, C. S., Zabusky; N. J. J Math Phys 1970, 11, 952-960.

[24] Marsden, J. E.; Tomba, A. J. Vector Calculus, 3rd ed.; W. H. Freeman and Company: New York, 1988.

[25] Newell, A. C. J Appl Mech 1983, 50, 1127-1137.

[26] Olver, P. J. Applications of Lie Groups to Differential Equations, 2nd ed.; Springer Verlag: New York, 1993.

[27] Sanz-Serna, J. M. J Comput Phys 1982, 47, 199-210.

[28] Svendsen, M.; Fogedby, H. C. J Phys A: Math Gen 1993, 26, 1717-1730.

[29] Wolf, T. Euro J Appl Math 2002, 13, 129-152.

[30] Wolf, T.; Brand, A.; Mohammadzadeh, M. J Symb Comp 1999, 27, 221-238.